Algebraic characterization of dendricity

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LAMFA

Main result

Theorem (Berthé, De Felice, Dolce, Leroy, Perrin, Reutenauer, Rindone & G., Goulet-Ouellet, Leroy, Stas)

Let X be a minimal shift space over $\mathcal{A}$. The following assertions are equivalent:

1. *X is dendric;*
2. *for each $w \in \mathcal{L}(X)$, $\mathcal{R}_X(w)$ is a basis of $F_{\mathcal{A}}$.*

The protagonists: (unidimensional) minimal shift spaces

Let $\mathcal{A}$ be a finite set called *alphabet*.

Definition

A shift space X is a

- closed subset of $\mathcal{A}^{\mathbb{Z}}$
- stable under the shift map $S : (x_i)_{i \in \mathbb{Z}} \mapsto (x_{i+1})_{i \in \mathbb{Z}}$.

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Definition

The *language* of a shift space X is

$$\mathcal{L}(X) = \{w : \exists x \in X, \exists\ i \leq j \text{ st. } w = x_i \cdots x_j\}.$$

The protagonists: dendricity (1)

$$\cdots 10010011001001001101100 \cdots$$

$$\mathcal{E}_X(10)$$

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$\cdots 100100110010010011011 00 \cdots$

$\mathcal{E}_X(10)$

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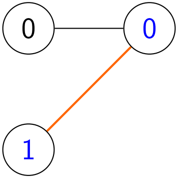
$\cdots 1001001100100100 1101100 \cdots$

$\mathcal{E}_X(10)$



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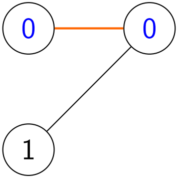
$$\cdots 100100110\,0\,1\,0\,0\,1001101100 \cdots$$

$\mathcal{E}_X(10)$



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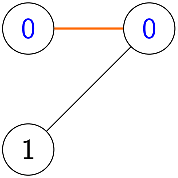

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$$\cdots 1001001100100100\textcolor{blue}{1}\textcolor{orange}{10}\textcolor{blue}{1}100 \cdots$$

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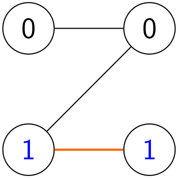

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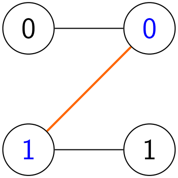
$\mathcal{E}_X(10)$



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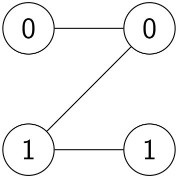

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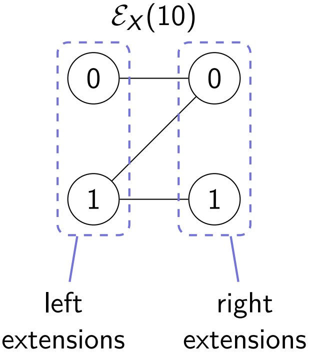

$$E_X^L(w) = \{a \in \mathcal{A} : aw \in \mathcal{L}(X)\} \qquad E_X^R(w) = \{b \in \mathcal{A} : wb \in \mathcal{L}(X)\}$$

$$E_X(w) = \{(a,b) \in \mathcal{A} \times \mathcal{A} : awb \in \mathcal{L}(X)\}$$

The protagonists: dendricity (2)

Definition (Berthé *et al.*)

A word $w \in \mathcal{L}(X)$ is *dendric* (resp., *connected*) if its extension graph is a tree (resp., is connected).

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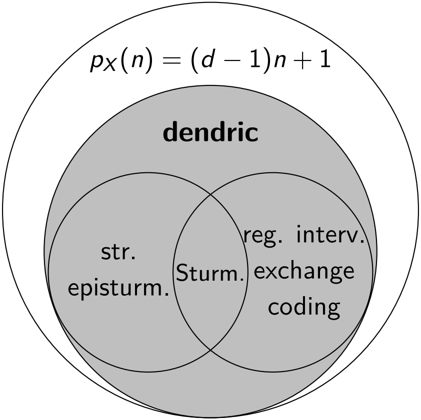
A shift space X is *dendric* if every $w \in \mathcal{L}(X)$ is dendric.



The protagonists: return words

$$\cdots 0011002001100220011000011 \cdots$$

The return words to 00 are:

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$$\cdots \mid 0011002001100220011000 11 \cdots$$

The return words to 00 are:

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$$\cdots \mid 0011 \mid 00200110022001100011 \cdots$$

The return words to 00 are: 0011

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$$\cdots \mid 0011 \mid 002 \mid 0011002200110001\!1 \cdots$$

The return words to 00 are: 0011, 002

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$$\cdots \mid 0011 \mid 002 \mid 0011 \mid 0022001100011 \cdots$$

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Definition

A *return word* for w is a word u such that

$$uw \in \mathcal{L}(X) \cap w\mathcal{A}^* \setminus \mathcal{A}^+ w \mathcal{A}^+.$$

The *set of return words* for w is denoted $\mathcal{R}_X(w)$.

The protagonists: return words

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For the example above,

$$\langle 0011, 002, 0022, 0 \rangle = \langle 0011, 002, 2, 0 \rangle = \langle 0011, 2, 0 \rangle = \langle 11, 2, 0 \rangle$$

Tool 1 : NEUTRALITY

Definition

Let

$$m_X(w) = \#E_X(w) - \#E_X^L(w) - \#E_X^R(w) + 1.$$

Definition

A word $w \in \mathcal{L}(X)$ is

- *neutral* if $m_X(w) = 0$;
- *weak* if $m_X(w) < 0$;
- *strong* if $m_X(w) > 0$.

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- *neutral* if $m_X(w) = 0$;
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Lemma

Let $w \in \mathcal{L}(X)$.

1. *If w is dendric, then w is neutral.*
2. *If w is connected, then w is NOT weak.*
3. *If w is connected and neutral, then w is dendric.*

Link with return words

Theorem (Balková, Pelantová, Steiner)

Let X be a minimal shift space with no weak $w \in \mathcal{L}(X)$. Then $\#\mathcal{R}_X(w) = \#\mathcal{A}$ for every $w \in \mathcal{L}(X)$ if and only if every $w \in \mathcal{L}(X)$ is neutral.

Idea of the proof: build a tree rooted in w in which we see (complete) return words as leafs, then use the link between edges and vertices in a tree.

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Idea of the proof: build a tree rooted in w in which we see (complete) return words as leafs, then use the link between edges and vertices in a tree.

What we still need to prove:

- if X is dendric, then $\mathcal{R}_X(w)$ generates $F_\mathcal{A}$ for every $w \in \mathcal{L}(X)$;
- if $\mathcal{R}_X(w)$ is a basis of $F_\mathcal{A}$ for every $w \in \mathcal{L}(X)$, then every $w \in \mathcal{L}(X)$ is connected.

Tool 2 : RAUZY GRAPHS

Definition

Definition

Let X be a shift space. The *Rauzy graph of order n* is the graph $\Gamma_X(n)$ such that

- the vertices are the elements of $\mathcal{L}(X) \cap \mathcal{A}^n$;
- there is an edge from u to v labeled by $a \in \mathcal{A}$ if $av \in u\mathcal{A} \cap \mathcal{L}(X)$.

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$$\cdots 010020010020 \cdots$$

10 01

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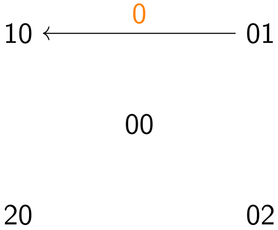

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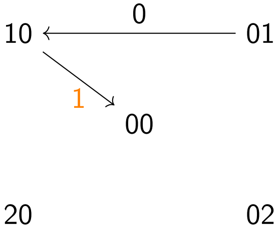

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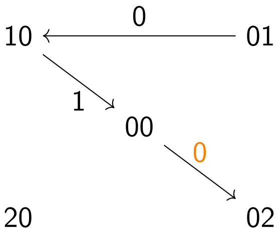

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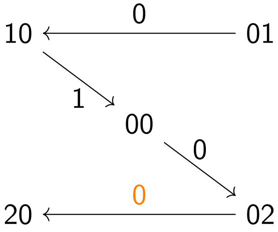

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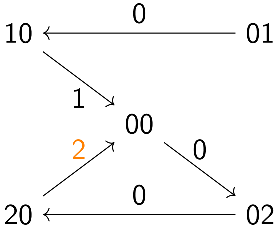

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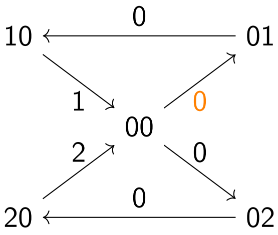

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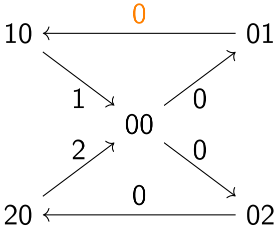

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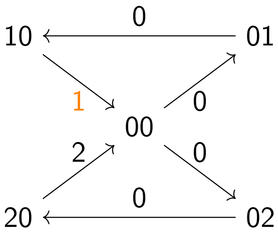

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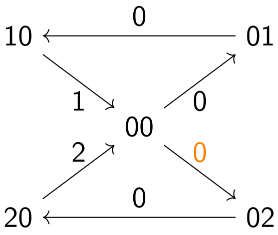

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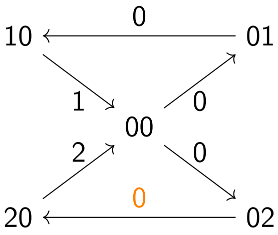

Link with return words

Return words for w are particular paths from w to w in $\Gamma_X(|w|)$.

Definition

The *Rauzy group* $G_X(w)$ associated with $w \in \mathcal{L}(X)$ is the subgroup of $F_{\mathcal{A}}$ generated by the labels of the paths from w to w in $\Gamma_X(|w|)$.

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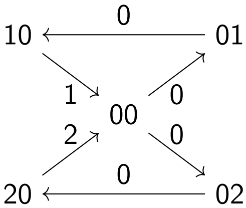
Proposition (Berthé *et al.*)

Let X be a minimal shift space and let $w \in \mathcal{L}(X)$. There exists $u \in \mathcal{R}_X(w)$ such that

$$G_X(uw) \leq \langle \mathcal{R}_X(w) \rangle \leq G_X(w).$$

Idea of the proof: a path from uw to uw is a concatenation of elements of $\mathcal{R}_{\mathcal{A}^{\mathbb{Z}}}(w)$, and if u is long enough, they are in $\mathcal{R}_X(w)$.

Dendricity and graph evolution



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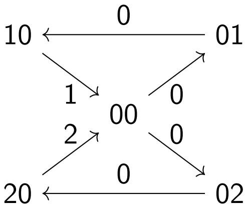

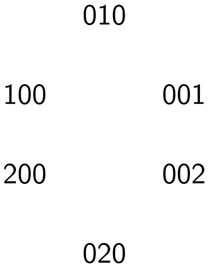

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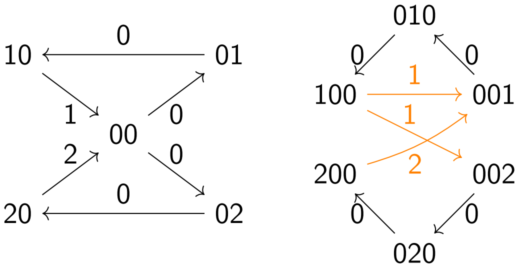

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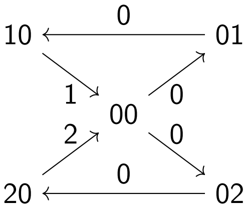

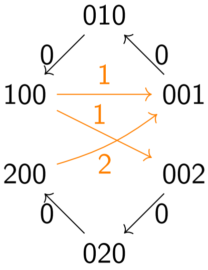

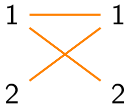

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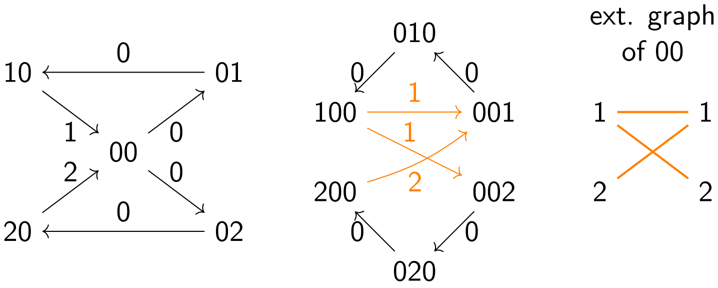

Proposition (Berthé *et al.*)

Let X be a minimal shift space. If every $w \in \mathcal{L}(X)$ is connected, then $G_X(w) = F_{\mathcal{A}}$ for every $w \in \mathcal{L}(X)$.

Idea of the proof: we can go from $\Gamma_X(n)$ to $\Gamma_X(n-1)$ using Stalling foldings, which preserve the Rauzy group.

Problem for the converse

In general,

$$\Gamma_X(n) \text{ and } \Gamma_X(n-1) \text{ ``generate'' the same group}$$
$$\not\Rightarrow \quad \text{every } w \in \mathcal{L}(X) \cap \mathcal{A}^{n-1} \text{ is connected}$$

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In general,

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BUT

Proposition (Goulet-Ouellet)

$$\Gamma_X(1) \textit{ generates } F_{\mathcal{A}} \Longrightarrow \varepsilon \textit{ is connected}$$

Idea of the proof: Only one type of Stallings folding is possible, and with the other one, we only merge connected right vertices.

Tool 3 : DERIVATION

Definition

$$\cdots \mid 0011 \mid 002 \mid 0011 \mid 0022 \mid 0011 \mid 0 \mid 0011 \cdots$$

Definition

$$\cdots \mid 0011 \mid 002 \mid 0011 \mid 0022 \mid 0011 \mid 0 \mid 0011 \cdots$$

$$\cdots \quad a \quad b \quad a \quad c \quad a \quad d \quad a \quad \cdots$$

Definition

$$\cdots \mid 0011 \mid 002 \mid 0011 \mid 0022 \mid 0011 \mid 0 \mid 0011 \cdots$$

$$\cdots \quad a \quad b \quad a \quad c \quad a \quad d \quad a \quad \cdots$$

Definition

Let X be minimal, $w \in \mathcal{L}(X)$ and $\sigma\colon \mathcal{B} \to \mathcal{R}_X(w)$ a bijection. The *derived shift space w.r.t w* is

$$D_w(X) = \{y \in \mathcal{B}^{\mathbb{Z}} : \cdots \sigma(y_{-1}).\sigma(y_0)\sigma(y_1)\cdots \in X\}.$$

Link with the main result

Lemma

Let X be a minimal shift space and let $w \in \mathcal{L}(X)$. If ε is connected in $D_w(X)$, then w is connected in X.

Idea of the proof: $ab \in \mathcal{L}(D_w(X))$ if and only if $\sigma(a)w\tilde{\sigma}(b) \in \mathcal{L}(X)$.

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Idea of the proof: $ab \in \mathcal{L}(D_w(X))$ if and only if $\sigma(a)w\tilde{\sigma}(b) \in \mathcal{L}(X)$.

Proposition (G., Goulet-Ouellet, Leroy, Stas)

Let X be a minimal shift space and let $w \in \mathcal{L}(X)$. If

- *$\mathcal{R}_X(w)$ is a basis of $\langle \mathcal{R}_X(w) \rangle$,*
- *there exists $u \in \mathcal{R}_X(w)$ such that $\langle \mathcal{R}_X(w) \rangle = \langle \mathcal{R}_X(uw) \rangle$,*

then $\Gamma_{D_w(X)}(1)$ generates $F_\mathcal{B}$.

Idea of the proof: if $\sigma(a) = u$, then $\langle \mathcal{R}_{D_w(X)}(a) \rangle = \sigma^{-1} \langle \mathcal{R}_X(uw) \rangle = F_\mathcal{B}$.

The end

Thank you for your attention!