Algebraic characterization of dendricity

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LAMFA

Main result

Theorem (Berthé, De Felice, Dolce, Leroy, Perrin, Reutenauer, Rindone & G., Goulet-Ouellet, Leroy, Stas)

Let X be a minimal shift space over $\mathcal{A}$. The following assertions are equivalent:

1. *X is dendric;*
2. *for each $w \in \mathcal{L}(X)$, $\mathcal{R}_X(w)$ is a basis of $F_{\mathcal{A}}$.*

The protagonists: (unidimensional) minimal shift spaces

Let $\mathcal{A}$ be a finite set called *alphabet*.

Definition

A shift space X is a

- closed subset of $\mathcal{A}^{\mathbb{Z}}$
- stable under the shift map $S : (x_i)_{i\in\mathbb{Z}} \mapsto (x_{i+1})_{i\in\mathbb{Z}}$.

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Definition

The *language* of a shift space X is

$$\mathcal{L}(X) = \{w : \exists x \in X, \exists\ i \leq j \text{ st. } w = x_i \cdots x_j\}.$$

The protagonists: dendricity (1)

$$\cdots 1001001100100100 1101100 \cdots$$

$$\mathcal{E}_X(10)$$

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$$\cdots 10010011001001001101100 \cdots$$

$\mathcal{E}_X(10)$

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$\cdots 1001001100 1001001101100 \cdots$

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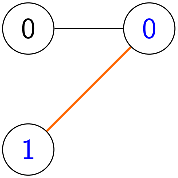

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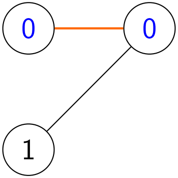

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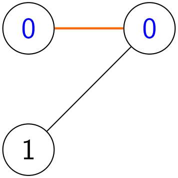

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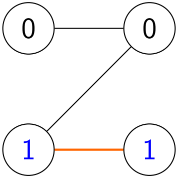

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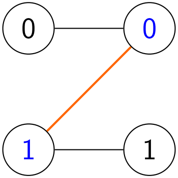

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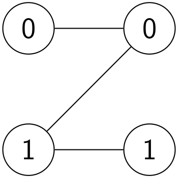

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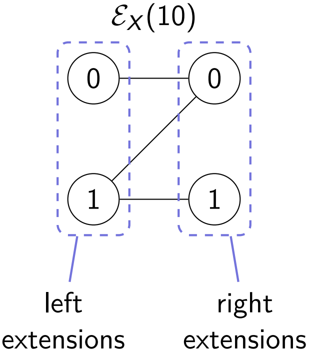

$$E_X^L(w) = \{a \in \mathcal{A} : aw \in \mathcal{L}(X)\} \qquad E_X^R(w) = \{b \in \mathcal{A} : wb \in \mathcal{L}(X)\}$$

$$E_X(w) = \{(a,b) \in \mathcal{A} \times \mathcal{A} : awb \in \mathcal{L}(X)\}$$

The protagonists: dendricity (2)

Definition (Berthé *et al.*)

A word $w \in \mathcal{L}(X)$ is *dendric* (resp., *connected*) if its extension graph is a tree (resp., is connected).

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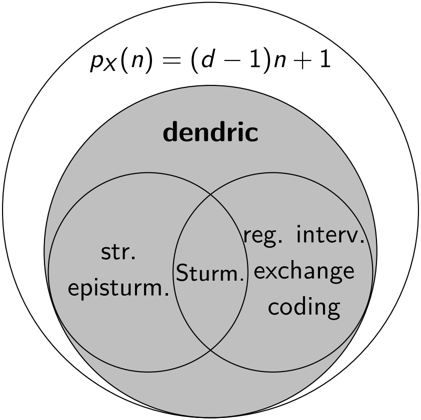
A shift space X is *dendric* if every $w \in \mathcal{L}(X)$ is dendric.



The protagonists: return words

$$\cdots 0011002001100220011000011 \cdots$$

The return words to 00 are:

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$$\cdots \mid 0011002001100220011000 11 \cdots$$

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$$\cdots \mid 0011 \mid 0020011002200110001 1\cdots$$

The return words to 00 are: 0011

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$$\cdots \mid 0011 \mid 002 \mid 0011002200110001 1\cdots$$

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Definition

A *return word* for w is a word u such that

$$uw \in \mathcal{L}(X) \cap w\mathcal{A}^+ \setminus \mathcal{A}^+ w \mathcal{A}^+.$$

The *set of return words* for w is denoted $\mathcal{R}_X(w)$.

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For the example above,

$$\langle 0011, 002, 0022, 0 \rangle = \langle 0011, 002, 2, 0 \rangle = \langle 0011, 2, 0 \rangle = \langle 11, 2, 0 \rangle$$

Main result (again)

Theorem (Berthé, De Felice, Dolce, Leroy, Perrin, Reutenauer, Rindone & G., Goulet-Ouellet, Leroy, Stas)

Let X be a minimal shift space over $\mathcal{A}$. The following assertions are equivalent:

1. *X is dendric;*
2. *for each $w \in \mathcal{L}(X)$, $\mathcal{R}_X(w)$ is a basis of $F_\mathcal{A}$;*
3. *for each $w \in \mathcal{L}(X)$, $\#\mathcal{R}_X(w) = \#\mathcal{A}$ and $\langle \mathcal{R}_X(w) \rangle = F_\mathcal{A}$.*

Tool 1 : NEUTRALITY

Definition

Let

$$m_X(w) = \#E_X(w) - \#E_X^L(w) - \#E_X^R(w) + 1.$$

Definition

A word $w \in \mathcal{L}(X)$ is

- *neutral* if $m_X(w) = 0$;
- *weak* if $m_X(w) < 0$;
- *strong* if $m_X(w) > 0$.

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- *neutral* if $m_X(w) = 0$;
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Lemma

Let $w \in \mathcal{L}(X)$.

1. *If w is dendric, then w is neutral.*
2. *If w is connected, then w is NOT weak.*
3. *If w is connected and neutral, then w is dendric.*

Link with return words

Theorem (Balková, Pelantová, Steiner)

Let X be a minimal shift space with no weak $w \in \mathcal{L}(X)$. The following are equivalent:

1. $\#\mathcal{R}_X(w) = \#\mathcal{A}$ *for every* $w \in \mathcal{L}(X)$*;*
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$$\#\mathcal{R}_X(w) = \#\mathcal{A} + \sum_{u \in T_w} m_X(u) \text{ for some } T_w,$$

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- conclude as there are no weak words.

Link with the main result

This result shows:

- If X is dendric, then $\#\mathcal{R}_X(w) = \#\mathcal{A}$ for each $w \in \mathcal{L}(X)$.

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- If X is dendric, then $\langle \mathcal{R}_X(w) \rangle = F_{\mathcal{A}}$ for each $w \in \mathcal{L}(X)$.
- If $\mathcal{R}_X(w)$ is a basis of $F_{\mathcal{A}}$ for every $w \in \mathcal{L}(X)$, then every $w \in \mathcal{L}(X)$ is connected.

Tool 2 : RAUZY GRAPHS

Definition

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Let X be a shift space. The *Rauzy graph of order n* is the graph $\Gamma_X(n)$ such that

- the vertices are the elements of $\mathcal{L}(X) \cap \mathcal{A}^n$;
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10 01

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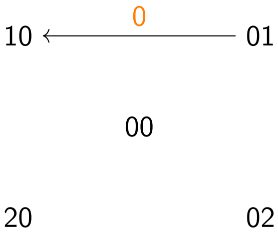

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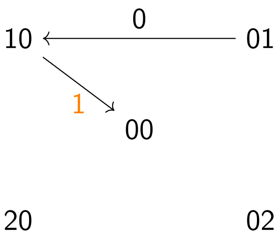

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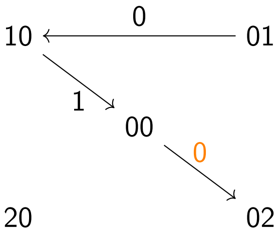

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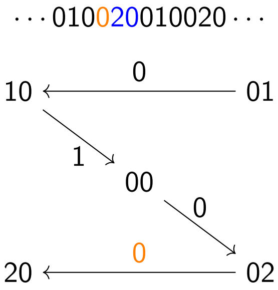

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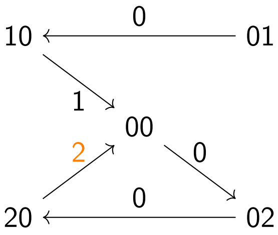

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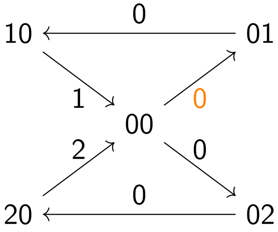

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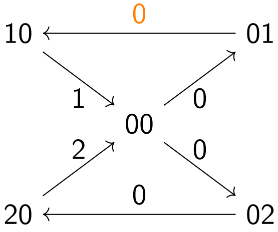

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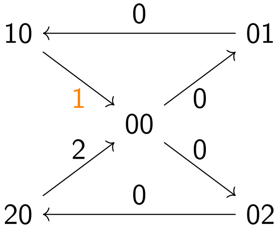

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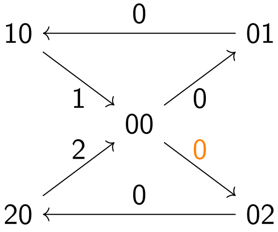

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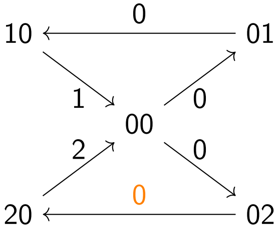

Link with return words

Return words for w are particular paths from w to w in $\Gamma_X(|w|)$.

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The *Rauzy group* $G_X(w)$ associated with $w \in \mathcal{L}(X)$ is the subgroup of $F_\mathcal{A}$ generated by the labels of the paths from w to w in $\Gamma_X(|w|)$.

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Clearly,

$$\langle \mathcal{R}_X(w) \rangle \leq G_X(w).$$

Nested groups

Proposition (Berthé *et al.*)

Let X be a minimal shift space and let $w \in \mathcal{L}(X)$. There exists $u \in \mathcal{R}_X(w)$ such that

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- if $|v_i w| \leq |uw|$, then $v_i w$ is in $\mathcal{L}(X)$ so $v_i \in \mathcal{R}_X(w)$,

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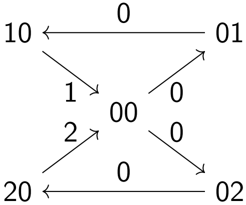
Idea of the proof:

- take u such that $|u| = \max(|\mathcal{R}_X(w)|)$,
- as uw starts with w, a path from uw to uw is a concatenation of words $v_i \in \mathcal{R}_{\mathcal{A}^{\mathbb{Z}}}(w)$,
- $v_i w$ labels a path in $\Gamma_X(|uw|)$ so its length-($\min(|uw|, |v_i w|)$) prefix is a factor of a vertex and is in $\mathcal{L}(X)$,
- if $|v_i w| \leq |uw|$, then $v_i w$ is in $\mathcal{L}(X)$ so $v_i \in \mathcal{R}_X(w)$,
- otherwise, by definition of u, w is an internal factor of $v_i w$, a contradiction.

Dendricity and graph evolution



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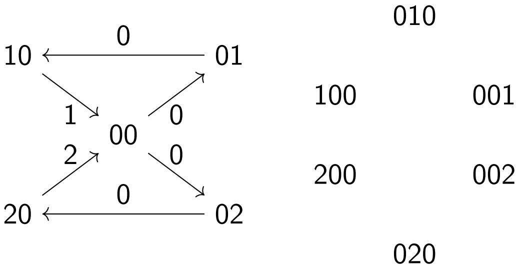

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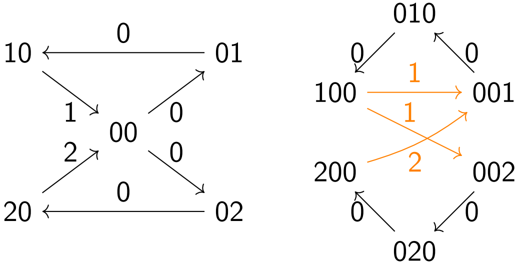

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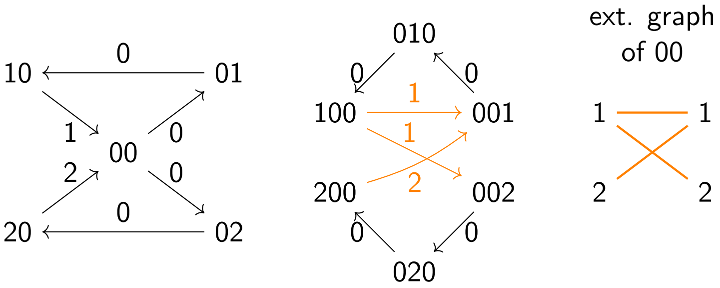

Link with the main result

Proposition (Berthé *et al.*)

Let X be a minimal shift space. If every $w \in \mathcal{L}(X)$ is connected, then $G_X(w) = F_{\mathcal{A}}$ for every $w \in \mathcal{L}(X)$.

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This ends the proof that, if X is dendric, then, for each $w \in \mathcal{L}(X)$, $\mathcal{R}_X(w)$ is a basis of $F_{\mathcal{A}}$.

Problem for the converse

In general,

$$\Gamma_X(n) \text{ and } \Gamma_X(n-1) \text{ "generate" the same group}$$
$$\not\Rightarrow \quad \text{every } w \in \mathcal{L}(X) \cap \mathcal{A}^{n-1} \text{ is connected}$$

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and

$$\langle \mathcal{R}_X(w) \rangle = F_{\mathcal{A}} \text{ for each } w \in \mathcal{L}(X) \cap \mathcal{A}^{\leq n}$$
$$\not\Rightarrow \quad \text{every } w \in \mathcal{L}(X) \cap \mathcal{A}^{\leq n-1} \text{ is connected}$$

Small hope

The implication is true for $n = 1$:

Proposition (Goulet-Ouellet)

Let X be a minimal shift space. If there exists $a \in \mathcal{A}$ such that $\langle \mathcal{R}_X(a) \rangle = F_{\mathcal{A}}$, then ε is connected.

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- in $\Gamma_X(1)$, edges labeled by the same letter leave the same vertex so only one type of Stallings foldings is possible,
- we only identify letters whose right instances are connected by a path in the extension graph of ε.

Tool 3 : DERIVATION

Definition

$$\cdots \mid 0011 \mid 002 \mid 0011 \mid 0022 \mid 0011 \mid 0 \mid 0011 \cdots$$

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Definition

Let X be minimal, $w \in \mathcal{L}(X)$ and $\sigma \colon \mathcal{B} \to \mathcal{R}_X(w)$ a bijection.
The *derived shift space w.r.t w* is

$$D_w(X) = \{y \in \mathcal{B}^{\mathbb{Z}} : \cdots \sigma(y_{-1}).\sigma(y_0)\sigma(y_1) \cdots \in X\}.$$

Link with connectedness

Lemma

Let X be a minimal shift space and let $w \in \mathcal{L}(X)$. If ε is connected in $D_w(X)$, then w is connected in X.

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- if $\sigma : \mathcal{B} \to \mathcal{R}_X(w)$ defines $D_w(X)$, set $\tilde{\sigma} : \mathcal{B} \to \mathcal{A}^*$ such that $\sigma(b)w = w\tilde{\sigma}(b)$,

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- $ab \in \mathcal{L}(D_w(X))$ if and only if $\sigma(a)w\tilde{\sigma}(b) \in \mathcal{L}(X)$,
- the extension graph of w in X is the image of the extension graph of ε in $D_w(X)$ under the graph morphism mapping left vertices a to the last letter of $\sigma(a)$ and right vertices b to the first letter of $\tilde{\sigma}(b)$.

Link with return words

Proposition (G., Goulet-Ouellet, Leroy, Stas)

Let X be a minimal shift space and let $w \in \mathcal{L}(X)$. If

- *$\mathcal{R}_X(w)$ is a basis of $\langle \mathcal{R}_X(w) \rangle$,*
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then there exists $a \in \mathcal{B}$ such that $\langle \mathcal{R}_{D_w(X)}(a) \rangle = F_{\mathcal{B}}$.

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- σ defines an isomorphism between $F_{\mathcal{B}}$ and $\langle \mathcal{R}_X(w) \rangle$,
- $\langle \mathcal{R}_{D_w(X)}(a) \rangle = \sigma^{-1} \langle \mathcal{R}_X(w) \rangle = F_{\mathcal{B}}$.

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- for every w, ε is connected in $D_w(X)$,
- every w is connected in X,
- as the number of return words is constant equal to $\#A$, X is dendric.

What's next?

- What about eventual dendricity?
- Can we characterize combinatorially the fact that $\langle \mathcal{R}_X(w) \rangle = F_{\mathcal{A}}$ for every $w \in \mathcal{L}(X)$?
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Thank you for your attention!