Decidability of dendricity (through a construction of Durand)

France Gheeraert
based on a joint work with Julien Leroy

June 2025

LAMFA

1. Introduction

Some basic definitions

- A letter is an element from a finite set (called alphabet).

Example:

$$210210221021021022022210210\cdots$$

Some basic definitions

- A letter is an element from a finite set (called alphabet).
- A word is a sequence of letters, it can be finite or infinite.

Example:

$$210210221021021022022210210\cdots$$

Some basic definitions

- A letter is an element from a finite set (called alphabet).
- A word is a sequence of letters, it can be finite or infinite.
- A finite word is a factor of a longer word if it appears consecutively in it.

Example:

$$210210221021021022022210210\cdots$$

Some basic definitions

- A letter is an element from a finite set (called alphabet).
- A word is a sequence of letters, it can be finite or infinite.
- A finite word is a factor of a longer word if it appears consecutively in it.
- The factor complexity of an infinite word is the function $p\colon \mathbb{N} \to \mathbb{N}$ counting, for each n, the number of different factors of length n.

Example:

$$2102102210210210220221021 0\cdots$$

Sturmian words

They have many definitions:

- aperiodic words of complexity $n + 1$ (the lowest possible)

Sturmian words

They have many definitions:

- aperiodic words of complexity $n+1$ (the lowest possible)
- aperiodic binary balanced words

Sturmian words

They have many definitions:

- aperiodic words of complexity $n + 1$ (the lowest possible)
- aperiodic binary balanced words
- codings of irrational rotation on the circle

Sturmian words

They have many definitions:

- aperiodic words of complexity $n + 1$ (the lowest possible)
- aperiodic binary balanced words
- codings of irrational rotation on the circle
- mechanical words
- ...

Sturmian words

They have many definitions:

- aperiodic words of complexity $n+1$ (the lowest possible)
- aperiodic binary balanced words
- codings of irrational rotation on the circle
- mechanical words
- ...

and many generalizations:

Sturmian words

They have many definitions:

- aperiodic words of complexity $n+1$ (the lowest possible)
- aperiodic binary balanced words
- codings of irrational rotation on the circle
- mechanical words
- ...

and many generalizations:

- Arnoux-Rauzy or episturmian words

Sturmian words

They have many definitions:

- aperiodic words of complexity $n + 1$ (the lowest possible)
- aperiodic binary balanced words
- codings of irrational rotation on the circle
- mechanical words
- ...

and many generalizations:

- Arnoux-Rauzy or episturmian words
- billiard words

Sturmian words

They have many definitions:

- aperiodic words of complexity $n+1$ (the lowest possible)
- aperiodic binary balanced words
- codings of irrational rotation on the circle
- mechanical words
- ...

and many generalizations:

- Arnoux-Rauzy or episturmian words
- billiard words
- codings of interval exchanges on the circle

Sturmian words

They have many definitions:

- aperiodic words of complexity $n + 1$ (the lowest possible)
- aperiodic binary balanced words
- codings of irrational rotation on the circle
- mechanical words
- ...

and many generalizations:

- Arnoux-Rauzy or episturmian words
- billiard words
- codings of interval exchanges on the circle
- **dendric words**
- ...

Extension graphs

$$x = 0010011001001001101100\cdots$$

$$\mathcal{E}_x(10)$$

Extension graphs

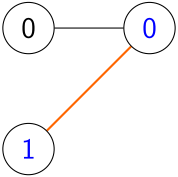
$$x = 0\color{blue}{0}\color{orange}{10}\color{blue}{0}\color{black}{1100100100110110}0\cdots$$

$$\mathcal{E}_x(10)$$

Extension graphs

$$x = 0010011001001001101100\cdots$$

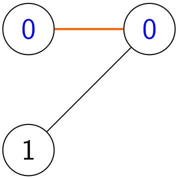
$\mathcal{E}_x(10)$



Extension graphs

$$x = 00100110\color{blue}{0}\color{orange}{10}\color{blue}{0}\color{black}{1001101100} \cdots$$

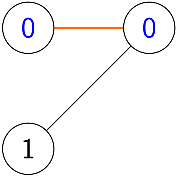
$\mathcal{E}_x(10)$



Extension graphs

$$x = 001001100100100 1101100 \cdots$$

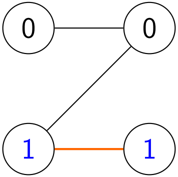
$\mathcal{E}_x(10)$



Extension graphs

$$x = 00100110010010011101100\cdots$$

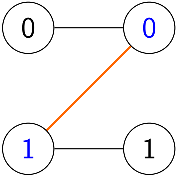
$\mathcal{E}_x(10)$



Extension graphs

$$x = 0010011001001001101\textcolor{blue}{1}\textcolor{orange}{10}\textcolor{blue}{0}\cdots$$

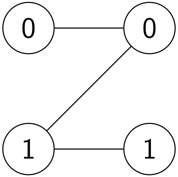
$\mathcal{E}_x(10)$



Extension graphs

$$x = 00100110010010011011100\cdots$$

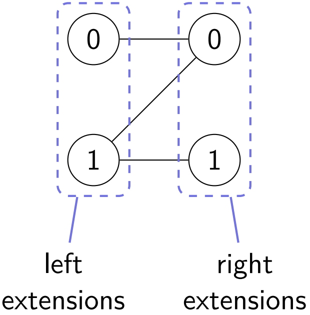
$$\mathcal{E}_x(10)$$



Extension graphs

$$x = 0010011001001001101100\cdots$$

$\mathcal{E}_x(10)$



A word is *bispecial* if it has at least 2 left extensions and 2 right extensions.

Dendricity

Definition (Berthé *et al.*)

A factor w of x is *dendric* if its extension graph is a tree.

Dendricity

Definition (Berthé *et al.*)

A factor w of x is *dendric* if its extension graph is a tree.

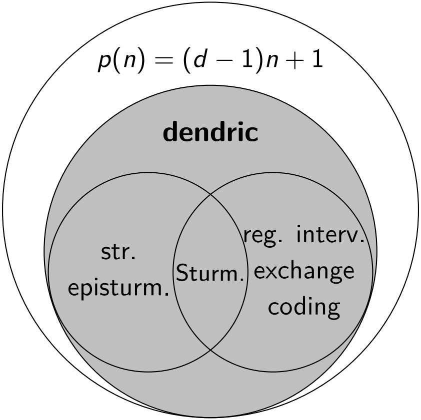
An infinite word is *dendric* if all of its factors are dendric.

Dendricity

Definition (Berthé *et al.*)

A factor w of x is *dendric* if its extension graph is a tree.

An infinite word is *dendric* if all of its factors are dendric.



Eventual dendricity

Definition (Dolce, Perrin)

An infinite word is *eventually dendric* if all but finitely many of its factors are dendric.

Eventual dendricity

Definition (Dolce, Perrin)

An infinite word is *eventually dendric* if all but finitely many of its factors are dendric. The *threshold* is the smallest N such that all factors of length at least N are dendric.

Eventual dendricity

Definition (Dolce, Perrin)

An infinite word is *eventually dendric* if all but finitely many of its factors are dendric. The *threshold* is the smallest N such that all factors of length at least N are dendric.

- includes dendric words, but also periodic or 1-balanced words

Eventual dendricity

Definition (Dolce, Perrin)

An infinite word is *eventually dendric* if all but finitely many of its factors are dendric. The *threshold* is the smallest N such that all factors of length at least N are dendric.

- includes dendric words, but also periodic or 1-balanced words
- eventually affine factor complexity

Eventual dendricity

Definition (Dolce, Perrin)

An infinite word is *eventually dendric* if all but finitely many of its factors are dendric. The *threshold* is the smallest N such that all factors of length at least N are dendric.

- includes dendric words, but also periodic or 1-balanced words
- eventually affine factor complexity
- dynamical property (invariant under topological isomorphism)

Eventual dendricity

Definition (Dolce, Perrin)

An infinite word is *eventually dendric* if all but finitely many of its factors are dendric. The *threshold* is the smallest N such that all factors of length at least N are dendric.

- includes dendric words, but also periodic or 1-balanced words
- eventually affine factor complexity
- dynamical property (invariant under topological isomorphism)
- admits several definitions

Eventual dendricity

Definition (Dolce, Perrin)

An infinite word is *eventually dendric* if all but finitely many of its factors are dendric. The *threshold* is the smallest N such that all factors of length at least N are dendric.

- includes dendric words, but also periodic or 1-balanced words
- eventually affine factor complexity
- dynamical property (invariant under topological isomorphism)
- admits several definitions
- stable under application of any morphism (Espinoza)

Two natural questions

1. How can we determine if a given word is (eventually) dendric?

Two natural questions

1. How can we determine if a given word is (eventually) dendric?
2. How can we generate “arbitrary” examples?

Two natural questions

1. How can we determine if a given word is (eventually) dendric?
2. How can we generate “arbitrary” examples?

We consider morphic words: $x = \tau(\sigma^\omega(a))$ where τ is non-erasing and σ is primitive and prolongeable on a.

Let's focus on two examples

Take the words

$$\mathbf{x} = \boldsymbol{\sigma}^{\omega}(0) = 0120101201201012010120120\,1\cdots$$
$$\mathbf{y} = \boldsymbol{\tau}(\boldsymbol{\sigma}^{\omega}(0)) = 1101111011011110111101 1\cdots$$

where

$$\boldsymbol{\tau}\colon \begin{cases} 0 \mapsto 1 \\ 1 \mapsto 1 \\ 2 \mapsto 0 \end{cases} \quad \text{and} \quad \boldsymbol{\sigma}\colon \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 2 \\ 2 \mapsto 01 \end{cases}.$$

Let's focus on two examples

Take the words

$$\mathbf{x} = \sigma^\omega(0) = 01201012012010120101201201\cdots$$
$$\mathbf{y} = \tau(\sigma^\omega(0)) = 110111101101111011111011\cdots$$

where

$$\tau : \begin{cases} 0 \mapsto 1 \\ 1 \mapsto 1 \\ 2 \mapsto 0 \end{cases} \quad \text{and} \quad \sigma : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 2 \\ 2 \mapsto 01 \end{cases} .$$

- the empty word is not dendric in $\mathbf{x}$,

Let's focus on two examples

Take the words

$$\mathbf{x} = \boldsymbol{\sigma}^{\omega}(0) = 0120101201201012010120120 1\cdots$$
$$\mathbf{y} = \boldsymbol{\tau}(\boldsymbol{\sigma}^{\omega}(0)) = 11011110110111101111011\cdots$$

where

$$\boldsymbol{\tau} : \begin{cases} 0 \mapsto 1 \\ 1 \mapsto 1 \\ 2 \mapsto 0 \end{cases} \quad \text{and} \quad \boldsymbol{\sigma} : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 2 \\ 2 \mapsto 01 \end{cases} .$$

- the empty word is not dendric in $\mathbf{x}$,
- 11 is not dendric in $\mathbf{y}$

Let's focus on two examples

Take the words

$$\mathbf{x} = \boldsymbol{\sigma}^{\omega}(0) = 0120101201201012010120120 1\cdots$$
$$\mathbf{y} = \boldsymbol{\tau}(\boldsymbol{\sigma}^{\omega}(0)) = 110111101101111011110 11\cdots$$

where

$$\boldsymbol{\tau}: \begin{cases} 0 \mapsto 1 \\ 1 \mapsto 1 \\ 2 \mapsto 0 \end{cases} \quad \text{and} \quad \boldsymbol{\sigma}: \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 2 \\ 2 \mapsto 01 \end{cases}.$$

- the empty word is not dendric in $\mathbf{x}$,
- 11 is not dendric in $\mathbf{y}$... but are $\mathbf{x}$ and $\mathbf{y}$ eventually dendric?

2. Durand's construction

Derivation

$$012010120120101201010120120\cdots$$

Derivation

$$0120101201201012010120120\cdots$$

$$012 \quad 01 \quad 012 \quad 012 \quad 01 \quad 012 \quad 01 \quad 012 \quad 012 \quad 0\cdots$$

Derivation

$$0120101201201012010120120\cdots$$

$$\begin{array}{cccccccccc}
012 & 01 & 012 & 012 & 01 & 012 & 01 & 012 & 012 & 0\cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \cdots
\end{array}$$

Derivation

$$01201012012010120101201 20\cdots$$

$$\begin{array}{cccccccccc} 012 & 01 & 012 & 012 & 01 & 012 & 01 & 012 & 012 & 0\cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \cdots \end{array}$$

Definition

The derived sequence of x with respect to its first letter a is the unique sequence $D(x)$ such that there exists an injective morphism θ satisfying:

- for any letter b, $\theta(b) \in a(\mathcal{A} \setminus a)^*$,
- $x = \theta(D(x))$.

Derivation

$$012010120120101201012012 0\cdots$$

$$\begin{array}{cccccccccc} 012 & 01 & 012 & 012 & 01 & 012 & 01 & 012 & 012 & 0\cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \cdots \end{array}$$

Definition

The derived sequence of x with respect to its first letter a is the unique sequence $D(x)$ such that there exists an injective morphism θ satisfying:

- for any letter b, $\theta(b) \in a(\mathcal{A} \setminus a)^*$,
- $x = \theta(D(x))$.

Here, $\theta(0) = 012$, $\theta(1) = 01$.

Construction of S-adic representation

Iterating the derivation, we obtain

$$x = \theta_1(D(x))$$

Construction of S-adic representation

Iterating the derivation, we obtain

$$\begin{aligned} x &= \theta_1(D(x)) \\ &= \theta_1(\theta_2(D^{(2)}(x))) \end{aligned}$$

Construction of S-adic representation

Iterating the derivation, we obtain

$$\begin{aligned} x &= \theta_1(D(x)) \\ &= \theta_1(\theta_2(D^{(2)}(x))) \\ &= \theta_1 \dots \theta_n(D^{(n)}(x)) \end{aligned}$$

Construction of S-adic representation

Iterating the derivation, we obtain

$$\begin{aligned} x &= \theta_1(D(x)) \\ &= \theta_1(\theta_2(D^{(2)}(x))) \\ &= \theta_1 \dots \theta_n(D^{(n)}(x)) \end{aligned}$$

So, if we choose the θ_i such that $D^{(n)}$ starts with a 0,

$$x = \lim_{n \to \infty} \theta_1 \dots \theta_n(0)$$

This means that the sequence (θ_i) entirely determines x, it is an *S-adic representation*.

The case of morphic words

Proposition (Durand)

For any morphic word x,

1. *$D(x)$ is morphic,*
2. *θ and generating morphisms for $D(x)$ can be algorithmically computed.*

Technique for purely morphic words

To find $D(x)$ when $x = \sigma^{\omega}(a)$:

Technique for purely morphic words

To find $D(x)$ when $x = \sigma^\omega(a)$:

1. Find the first return word r_0 for a by looking at iterates of σ on a.

Technique for purely morphic words

To find $D(x)$ when $x = \sigma^\omega(a)$:

1. Find the first return word r_0 for a by looking at iterates of σ on a.
2. Compute $\sigma(r_0)$ and decompose it into return words $r_0, r_1, \ldots$ for a.

Technique for purely morphic words

To find $D(x)$ when $x = \sigma^\omega(a)$:

1. Find the first return word r_0 for a by looking at iterates of σ on a.
2. Compute $\sigma(r_0)$ and decompose it into return words $r_0, r_1, \ldots$ for a.
3. If new return words were found, go back to step 2. for each one.

Technique for purely morphic words

To find $D(x)$ when $x = \sigma^\omega(a)$:

1. Find the first return word r_0 for a by looking at iterates of σ on a.
2. Compute $\sigma(r_0)$ and decompose it into return words $r_0, r_1, \dots$ for a.
3. If new return words were found, go back to step 2. for each one.
4. Set $\theta : i \mapsto r_i$ and $\varphi : i \mapsto j_1 \cdots j_n$ if

$$\sigma(r_i) = r_{j_1} \cdots r_{j_n}.$$

Technique for purely morphic words

To find $D(x)$ when $x = \sigma^\omega(a)$:

1. Find the first return word r_0 for a by looking at iterates of σ on a.
2. Compute $\sigma(r_0)$ and decompose it into return words $r_0, r_1, \ldots$ for a.
3. If new return words were found, go back to step 2. for each one.
4. Set $\theta : i \mapsto r_i$ and $\varphi : i \mapsto j_1 \cdots j_n$ if

$$\sigma(r_i) = r_{j_1} \cdots r_{j_n}.$$

Then $D(x) = \varphi^\omega(0)$ and $x = \theta(D(x))$.

Technique for purely morphic words

To find $D(x)$ when $x = \sigma^\omega(a)$:

1. Find the first return word r_0 for a by looking at iterates of σ on a.
2. Compute $\sigma(r_0)$ and decompose it into return words $r_0, r_1, \ldots$ for a.
3. If new return words were found, go back to step 2. for each one.
4. Set $\theta : i \mapsto r_i$ and $\varphi : i \mapsto j_1 \cdots j_n$ if

$$\sigma(r_i) = r_{j_1} \cdots r_{j_n}.$$

Then $D(x) = \varphi^\omega(0)$ and $x = \theta(D(x))$.

Important: θ depends on x, φ depends on σ.

Technique with a coding τ

To find $D(x)$ when $x = \tau(\sigma^\omega(a))$, we do the same BUT:

- we don't look at return words for $\tau(a)$ in x but at return words for $\tau^{-1}(\tau(a))$ in $\sigma^\omega(a)$;

Technique with a coding τ

To find $D(x)$ when $x = \tau(\sigma^\omega(a))$, we do the same BUT:

- we don't look at return words for $\tau(a)$ in x but at return words for $\tau^{-1}(\tau(a))$ in $\sigma^\omega(a)$;
- we replace return words r_i by complete return words $R_i = (r_i, b_i)$ where $b \in \tau^{-1}(\tau(a))$ is the letter following r_i;

Technique with a coding τ

To find $D(x)$ when $x = \tau(\sigma^\omega(a))$, we do the same BUT:

- we don't look at return words for $\tau(a)$ in x but at return words for $\tau^{-1}(\tau(a))$ in $\sigma^\omega(a)$;
- we replace return words r_i by complete return words $R_i = (r_i, b_i)$ where $b \in \tau^{-1}(\tau(a))$ is the letter following r_i;
- we may need to consider a power of σ;

Technique with a coding τ

To find $D(x)$ when $x = \tau(\sigma^\omega(a))$, we do the same BUT:

- we don't look at return words for $\tau(a)$ in x but at return words for $\tau^{-1}(\tau(a))$ in $\sigma^\omega(a)$;
- we replace return words r_i by complete return words $R_i = (r_i, b_i)$ where $b \in \tau^{-1}(\tau(a))$ is the letter following r_i;
- we may need to consider a power of σ;
- we have a coding for $D(x)$ that identifies indices i, j if $\tau(r_i) = \tau(r_j)$.

Process iteration

From $x = \tau(\sigma^\omega(a))$, we can construct

- (ρ_i, φ_i) such that $D^{(i)}(x) = \rho_i(\varphi_i^\omega(0))$,

Process iteration

From $x = \tau(\sigma^\omega(a))$, we can construct

- (ρ_i, φ_i) such that $D^{(i)}(x) = \rho_i(\varphi_i^\omega(0))$,
- θ_i such that $D^{(i-1)}(x) = \theta_i(D^{(i)}(x))$.

Process iteration

From $x = \tau(\sigma^\omega(a))$, we can construct

- (ρ_i, φ_i) such that $D^{(i)}(x) = \rho_i(\varphi_i^\omega(0))$,
- θ_i such that $D^{(i-1)}(x) = \theta_i(D^{(i)}(x))$.

Proposition (Durand)

If x is morphic, then

1. *there exist $i < j$ such that $D^{(i)}(x) = D^{(j)}(x)$*
   $\rightarrow$ *the sequence (θ_i) is eventually periodic;*

Process iteration

From $x = \tau(\sigma^\omega(a))$, we can construct

- (ρ_i, φ_i) such that $D^{(i)}(x) = \rho_i(\varphi_i^\omega(0))$,
- θ_i such that $D^{(i-1)}(x) = \theta_i(D^{(i)}(x))$.

Proposition (Durand)

If x is morphic, then

1. *there exist $i < j$ such that $D^{(i)}(x) = D^{(j)}(x)$*
   $\rightarrow$ the sequence (θ_i) is eventually periodic;
2. *there exist $i < j$ such that $(\rho_i, \varphi_i) = (\rho_j, \varphi_j)$*
   $\rightarrow$ we can compute the pre-period and period of (θ_i).

Process iteration

From $x = \tau(\sigma^\omega(a))$, we can construct

- (ρ_i, φ_i) such that $D^{(i)}(x) = \rho_i(\varphi_i^\omega(0))$,
- θ_i such that $D^{(i-1)}(x) = \theta_i(D^{(i)}(x))$.

Proposition (Durand)

If x is morphic, then

1. *there exist $i < j$ such that $D^{(i)}(x) = D^{(j)}(x)$*
   $\rightarrow$ the sequence (θ_i) is eventually periodic;
2. *there exist $i < j$ such that $(\rho_i, \varphi_i) = (\rho_j, \varphi_j)$*
   $\rightarrow$ we can compute the pre-period and period of (θ_i).

So we can the compute the sequence (θ_i).

For the examples

$\mathbf{x} = \boldsymbol{\sigma}^{\omega}(0)$:

- $\theta_1\colon 0 \mapsto 012, 1 \mapsto 01$, $\varphi_1\colon 0 \mapsto 01, 1 \mapsto 0$

For the examples

$\mathbf{x} = \boldsymbol{\sigma}^{\omega}(0)$:

- $\theta_1 \colon 0 \mapsto 012, 1 \mapsto 01$, $\varphi_1 \colon 0 \mapsto 01, 1 \mapsto 0$
- $\theta_2 \colon 0 \mapsto 01, 1 \mapsto 0$, $\varphi_2 \colon 0 \mapsto 01, 1 \mapsto 0$

For the examples

$\mathbf{x} = \boldsymbol{\sigma}^{\omega}(0)$:

- $\theta_1 \colon 0 \mapsto 012, 1 \mapsto 01$, $\varphi_1 \colon 0 \mapsto 01, 1 \mapsto 0$
- $\theta_2 \colon 0 \mapsto 01, 1 \mapsto 0$, $\varphi_2 \colon 0 \mapsto 01, 1 \mapsto 0$

$\rightarrow$ $(\theta_i) = \theta_1\theta_2\theta_2\theta_2 \dots$

For the examples

$\mathbf{x} = \boldsymbol{\sigma}^\omega(0)$:

- $\theta_1 \colon 0 \mapsto 012, 1 \mapsto 01$, $\varphi_1 \colon 0 \mapsto 01, 1 \mapsto 0$
- $\theta_2 \colon 0 \mapsto 01, 1 \mapsto 0$, $\varphi_2 \colon 0 \mapsto 01, 1 \mapsto 0$

$\rightarrow$ $(\theta_i) = \theta_1\theta_2\theta_2\theta_2 \ldots$

$\mathbf{y} = \boldsymbol{\tau}(\boldsymbol{\sigma}^\omega(0))$:

- $\theta_1 \colon 0 \mapsto 1, 1 \mapsto 10$, $\rho_1 \colon 0, 2 \mapsto 0, 1 \mapsto 1$,
  $\varphi_1 \colon 0 \mapsto 01, 1 \mapsto 0201, 2 \mapsto 02$

For the examples

$\mathbf{x} = \boldsymbol{\sigma}^{\omega}(0)$:

- $\theta_1 \colon 0 \mapsto 012, 1 \mapsto 01$, $\varphi_1 \colon 0 \mapsto 01, 1 \mapsto 0$
- $\theta_2 \colon 0 \mapsto 01, 1 \mapsto 0$, $\varphi_2 \colon 0 \mapsto 01, 1 \mapsto 0$

$\rightarrow$ $(\theta_i) = \theta_1\theta_2\theta_2\theta_2 \dots$

$\mathbf{y} = \boldsymbol{\tau}(\boldsymbol{\sigma}^{\omega}(0))$:

- $\theta_1 \colon 0 \mapsto 1, 1 \mapsto 10$, $\rho_1 \colon 0, 2 \mapsto 0, 1 \mapsto 1$,
  $\varphi_1 \colon 0 \mapsto 01, 1 \mapsto 0201, 2 \mapsto 02$
- $\theta_2 \colon 0 \mapsto 01, 1 \mapsto 0$, $\rho_2 \colon 0 \mapsto 0, 1, 2 \mapsto 1$,
  $\varphi_2 \colon 0 \mapsto 0120, 1 \mapsto 0, 2 \mapsto 12$

For the examples

$\mathbf{x} = \boldsymbol{\sigma}^{\omega}(0)$:

- $\theta_1 \colon 0 \mapsto 012, 1 \mapsto 01$, $\varphi_1 \colon 0 \mapsto 01, 1 \mapsto 0$
- $\theta_2 \colon 0 \mapsto 01, 1 \mapsto 0$, $\varphi_2 \colon 0 \mapsto 01, 1 \mapsto 0$

$\rightarrow$ $(\theta_i) = \theta_1\theta_2\theta_2\theta_2 \ldots$

$\mathbf{y} = \boldsymbol{\tau}(\boldsymbol{\sigma}^{\omega}(0))$:

- $\theta_1 \colon 0 \mapsto 1, 1 \mapsto 10$, $\rho_1 \colon 0, 2 \mapsto 0, 1 \mapsto 1$, $\varphi_1 \colon 0 \mapsto 01, 1 \mapsto 0201, 2 \mapsto 02$
- $\theta_2 \colon 0 \mapsto 01, 1 \mapsto 0$, $\rho_2 \colon 0 \mapsto 0, 1, 2 \mapsto 1$, $\varphi_2 \colon 0 \mapsto 0120, 1 \mapsto 0, 2 \mapsto 12$
- $\theta_3 \colon 0 \mapsto 011, 1 \mapsto 0$, $\varphi_3 \colon 0 \mapsto 010, 1 \mapsto 01$

For the examples

$\mathbf{x} = \boldsymbol{\sigma}^{\omega}(0)$:

- $\theta_1 \colon 0 \mapsto 012, 1 \mapsto 01$, $\varphi_1 \colon 0 \mapsto 01, 1 \mapsto 0$
- $\theta_2 \colon 0 \mapsto 01, 1 \mapsto 0$, $\varphi_2 \colon 0 \mapsto 01, 1 \mapsto 0$

$\rightarrow$ $(\theta_i) = \theta_1\theta_2\theta_2\theta_2\ldots$

$\mathbf{y} = \boldsymbol{\tau}(\boldsymbol{\sigma}^{\omega}(0))$:

- $\theta_1 \colon 0 \mapsto 1, 1 \mapsto 10$, $\rho_1 \colon 0, 2 \mapsto 0, 1 \mapsto 1$, $\varphi_1 \colon 0 \mapsto 01, 1 \mapsto 0201, 2 \mapsto 02$
- $\theta_2 \colon 0 \mapsto 01, 1 \mapsto 0$, $\rho_2 \colon 0 \mapsto 0, 1, 2 \mapsto 1$, $\varphi_2 \colon 0 \mapsto 0120, 1 \mapsto 0, 2 \mapsto 12$
- $\theta_3 \colon 0 \mapsto 011, 1 \mapsto 0$, $\varphi_3 \colon 0 \mapsto 010, 1 \mapsto 01$
- $\theta_4 \colon 0 \mapsto 01, 1 \mapsto 0$, $\varphi_4 \colon 0 \mapsto 010, 1 \mapsto 01$

For the examples

$\mathbf{x} = \boldsymbol{\sigma}^\omega(0)$:

- $\theta_1 \colon 0 \mapsto 012, 1 \mapsto 01$, $\varphi_1 \colon 0 \mapsto 01, 1 \mapsto 0$
- $\theta_2 \colon 0 \mapsto 01, 1 \mapsto 0$, $\varphi_2 \colon 0 \mapsto 01, 1 \mapsto 0$

$\rightarrow$ $(\theta_i) = \theta_1\theta_2\theta_2\theta_2 \dots$

$\mathbf{y} = \boldsymbol{\tau}(\boldsymbol{\sigma}^\omega(0))$:

- $\theta_1 \colon 0 \mapsto 1, 1 \mapsto 10$, $\rho_1 \colon 0, 2 \mapsto 0, 1 \mapsto 1$, $\varphi_1 \colon 0 \mapsto 01, 1 \mapsto 0201, 2 \mapsto 02$
- $\theta_2 \colon 0 \mapsto 01, 1 \mapsto 0$, $\rho_2 \colon 0 \mapsto 0, 1, 2 \mapsto 1$, $\varphi_2 \colon 0 \mapsto 0120, 1 \mapsto 0, 2 \mapsto 12$
- $\theta_3 \colon 0 \mapsto 011, 1 \mapsto 0$, $\varphi_3 \colon 0 \mapsto 010, 1 \mapsto 01$
- $\theta_4 \colon 0 \mapsto 01, 1 \mapsto 0$, $\varphi_4 \colon 0 \mapsto 010, 1 \mapsto 01$

$\rightarrow$ $(\theta_i) = \theta_1\theta_2\theta_3\theta_4\theta_4\theta_4 \dots$

Decidability of the equality

Corollary

If we can apply Durand's construction on $x = \tau_1(\sigma_1^\omega(a))$ and $y = \tau_2(\sigma_2^\omega(b))$, then we can decide whether $x = y$.

Decidability of the equality

Corollary

If we can apply Durand's construction on $x = \tau_1(\sigma_1^\omega(a))$ and $y = \tau_2(\sigma_2^\omega(b))$, then we can decide whether $x = y$.

Proof:
It suffices to check whether the associated sequences (θ_i) are equal since they only depend on the initial word and not on its morphic representation.

Alternative morphic representation

$$
\begin{aligned}
x &= \theta_0 \cdots \theta_{i-1} \left(\theta_i \cdots \theta_{j-1}\right)^{\omega}(0) \\
&= \alpha(\beta^{\omega}(0))
\end{aligned}
$$

where α and β are return morphisms.

Alternative morphic representation

$$\begin{aligned} x &= \theta_0 \cdots \theta_{i-1} \left(\theta_i \cdots \theta_{j-1}\right)^{\omega}(0) \\ &= \alpha(\beta^{\omega}(0)) \end{aligned}$$

where α and β are return morphisms.

Definition

A morphism α is a return morphism if there exists a word w such that

- α is injective over the letters
- for any letter a , $\alpha(a)w$ starts with w and contains no internal occurrence of w.

Alternative morphic representation

$$\mathbf{y} = \theta_0 \cdots \theta_{i-1} \left(\theta_i \cdots \theta_{j-1}\right)^{\omega}(0)$$
$$= \boldsymbol{\alpha}(\boldsymbol{\beta}^{\omega}(0))$$

where α and β are return morphisms.

$$\boldsymbol{\alpha} : \begin{cases} 0 \mapsto 11011110 \\ 1 \mapsto 110110 \end{cases}$$

$$\boldsymbol{\beta} : \begin{cases} 0 \mapsto 010 \\ 1 \mapsto 00 \end{cases}$$

Definition

A morphism α is a return morphism if there exists a word w such that

- α is injective over the letters
- for any letter a , $\alpha(a)w$ starts with w and contains no internal occurrence of w.

Description of extensions

u in $\alpha(\beta^\omega(0))$ 1110110111101111011

Description of extensions

u in $\alpha(\beta^\omega(0))$ $\quad a11101101111011110 11b$

Description of extensions

$$a1 \mid 110 \mid 11011 \mid 11011 \mid 11011b$$

$$\uparrow$$

u in $\boldsymbol{\alpha}(\boldsymbol{\beta}^\omega(0))$ $\qquad a11101101111011111011b$

$$\uparrow \boldsymbol{\alpha}$$

v in $\boldsymbol{\beta}^\omega(0)$ $\qquad ?$

Description of extensions

suffix of $\alpha(c)$ prefix of $\alpha(d)110$

$a1 \mid 110 \mid 11011 \mid 11011 \mid 11011b$

u in $\alpha(\beta^\omega(0))$ $a11101101111011111011b$

α

v in $\beta^\omega(0)$?

Description of extensions

suffix of $\alpha(c)$ prefix of $\alpha(d)110$

$a1 \mid 110 \mid 11011 \mid 11011 \mid 11011b$

u in $\alpha(\beta^\omega(0))$ $a11101101111011111011b$

α

v in $\beta^\omega(0)$ $c100d$

Description of extensions

suffix of $\alpha(c)$ prefix of $\alpha(d)110$

$\boxed{a1} \mid 110 \mid 11011 \mid 11011 \mid \boxed{11011b}$

u in $\alpha(\beta^\omega(0))$ $a11101101111011111011b$

$$c100d \xrightarrow{\alpha} a11101101111011111011b$$

v in $\beta^\omega(0)$ $c100d$

- v is the *parent* of u, and u is <u>an</u> *extended image* of v

Description of extensions

suffix of $\alpha(c)$ prefix of $\alpha(d)110$

$\boxed{a1} \mid 110 \mid 11011 \mid 11011 \mid \boxed{11011b}$

u in $\alpha(\beta^\omega(0))$ $a11101101111011111011b$

$\uparrow \alpha$

v in $\beta^\omega(0)$ $c100d$

- v is the *parent* of u, and u is <u>an</u> *extended image* of v
- the extensions of a parent entirely determine the extensions of its extended images

3. Back to dendricity

Eventual dendricity

Let $x = \theta(y)$ with θ a return morphism (for the word w).

Proposition (Berthé *et al.*, G., Leroy)

If x is eventually dendric of threshold N, then y is eventually dendric of threshold $\leq \max\{0, N - |w|\}$.

Eventual dendricity

Let $x = \theta(y)$ with θ a return morphism (for the word w).

Proposition (Berthé *et al.*, G., Leroy)

If x is eventually dendric of threshold N, then y is eventually dendric of threshold $\leq \max\{0, N - |w|\}$.

Proposition (G., Leroy)

$$x \text{ eventually dendric} \iff y \text{ eventually dendric}$$

Eventual dendricity

Let $x = \theta(y)$ with θ a return morphism (for the word w).

Proposition (Berthé *et al.*, G., Leroy)

If x is eventually dendric of threshold N, then y is eventually dendric of threshold $\leq \max\{0, N - |w|\}$.

Proposition (G., Leroy)

$$x \text{ eventually dendric} \iff y \text{ eventually dendric}$$

$$\mathbf{y} = \boldsymbol{\alpha}(\boldsymbol{\beta}^{\omega}(0)) \text{ eventually dendric}$$

Eventual dendricity

Let $x = \theta(y)$ with θ a return morphism (for the word w).

Proposition (Berthé *et al.*, G., Leroy)

If x is eventually dendric of threshold N, then y is eventually dendric of threshold $\leq \max\{0, N - |w|\}$.

Proposition (G., Leroy)

$$x \textit{ eventually dendric} \iff y \textit{ eventually dendric}$$

$$\begin{aligned} \mathbf{y} = \boldsymbol{\alpha}(\boldsymbol{\beta}^\omega(0)) &\text{ eventually dendric} \\ \iff \boldsymbol{\beta}^\omega(0) &\text{ eventually dendric} \end{aligned}$$

Eventual dendricity

Let $x = \theta(y)$ with θ a return morphism (for the word w).

Proposition (Berthé *et al.*, G., Leroy)

If x is eventually dendric of threshold N, then y is eventually dendric of threshold $\leq \max\{0, N - |w|\}$.

Proposition (G., Leroy)

$$x \text{ eventually dendric} \iff y \text{ eventually dendric}$$

$$\begin{aligned} \mathbf{y} = \boldsymbol{\alpha}(\boldsymbol{\beta}^{\omega}(0)) &\text{ eventually dendric} \\ \iff \boldsymbol{\beta}^{\omega}(0) &\text{ eventually dendric} \\ \iff \boldsymbol{\beta}^{\omega}(0) &\text{ dendric} \end{aligned}$$

Eventual dendricity

Let $x = \theta(y)$ with θ a return morphism (for the word w).

Proposition (Berthé *et al.*, G., Leroy)

If x is eventually dendric of threshold N, then y is eventually dendric of threshold $\leq \max\{0, N - |w|\}$.

Proposition (G., Leroy)

$$x \text{ eventually dendric} \iff y \text{ eventually dendric}$$

$$\begin{aligned} \mathbf{y} = \boldsymbol{\alpha}(\beta^\omega(0)) &\text{ eventually dendric} \\ \iff \beta^\omega(0) &\text{ eventually dendric} \\ \iff \beta^\omega(0) &\text{ dendric} \quad \checkmark \end{aligned}$$

A new example

We now consider $\mathbf{z} = \boldsymbol{\lambda}(\boldsymbol{\mu}^{\omega}(0))$ where

$$\boldsymbol{\lambda}\colon \begin{cases} 0 \mapsto 02 \\ 1 \mapsto 0 \\ 2 \mapsto 012 \end{cases} \quad \text{and} \quad \boldsymbol{\mu}\colon \begin{cases} 0 \mapsto 01002 \\ 1 \mapsto 0100201 \\ 2 \mapsto 01002010201 \end{cases}$$

A new example

We now consider $\mathbf{z} = \boldsymbol{\lambda}(\mu^\omega(0))$ where

$$\boldsymbol{\lambda}\colon \begin{cases} 0 \mapsto 02 \\ 1 \mapsto 0 \\ 2 \mapsto 012 \end{cases} \quad \text{and} \quad \mu\colon \begin{cases} 0 \mapsto 01002 \\ 1 \mapsto 0100201 \\ 2 \mapsto 01002010201 \end{cases}$$

We need to determine

- whether $\mu^\omega(0)$ is dendric
  $\rightarrow$ $\mathbf{z}$ is eventually dendric,

A new example

We now consider $\mathbf{z} = \boldsymbol{\lambda}(\boldsymbol{\mu}^\omega(0))$ where

$$\boldsymbol{\lambda} \colon \begin{cases} 0 \mapsto 02 \\ 1 \mapsto 0 \\ 2 \mapsto 012 \end{cases} \qquad \text{and} \qquad \boldsymbol{\mu} \colon \begin{cases} 0 \mapsto 01002 \\ 1 \mapsto 0100201 \\ 2 \mapsto 01002010201 \end{cases}$$

We need to determine

- whether $\boldsymbol{\mu}^\omega(0)$ is dendric
  $\rightarrow$ $\mathbf{z}$ is eventually dendric,
- and if yes, whether its image under $\boldsymbol{\lambda}$ is dendric
  $\rightarrow$ $\mathbf{z}$ is dendric.

Factor generation

We can partition the factors of $\mu^{\omega}(0)$ as follows:

- gen. 0: factors that do not have a parent
- gen. 1: factors whose parents are of gen. 0
- gen. n: factors whose parents are of gen. $n-1$

Factor generation

We can partition the factors of $\mu^\omega(0)$ as follows:

- gen. 0: factors that do not have a parent
  $=$ factors without occurrence of 0100
- gen. 1: factors whose parents are of gen. 0
- gen. n: factors whose parents are of gen. $n-1$

Factor generation

We can partition the factors of $\mu^\omega(0)$ as follows:

- gen. 0: factors that do not have a parent
  $= $ factors without occurrence of 0100
  $= $ internal factors of $\mu(a)0100$
- gen. 1: factors whose parents are of gen. 0
- gen. n: factors whose parents are of gen. $n-1$

Factor generation

We can partition the factors of $\mu^\omega(0)$ as follows:

- gen. 0: factors that do not have a parent
  $=$ factors without occurrence of 0100
  $=$ internal factors of $\mu(a)0100$
- gen. 1: factors whose parents are of gen. 0
- gen. n: factors whose parents are of gen. $n-1$

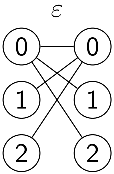
$$\mu : \begin{cases} 0 \mapsto 01002\textcolor{gray}{0100} \\ 1 \mapsto 0100201\textcolor{gray}{0100} \\ 2 \mapsto 0100201020 1\textcolor{gray}{0100} \end{cases}$$

Factor generation

We can partition the factors of $\mu^{\omega}(0)$ as follows:

- gen. 0: factors that do not have a parent
  $=$ factors without occurrence of 0100
  $=$ internal factors of $\mu(a)0100$
- gen. 1: factors whose parents are of gen. 0
- gen. n: factors whose parents are of gen. $n-1$

$$\mu : \begin{cases} 0 \mapsto 010020100 \\ 1 \mapsto 010020100100 \\ 2 \mapsto 010020102010100 \end{cases}$$

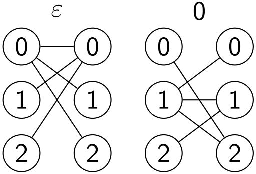


Factor generation

We can partition the factors of $\mu^\omega(0)$ as follows:

- gen. 0: factors that do not have a parent
  $=$ factors without occurrence of 0100
  $=$ internal factors of $\mu(a)0100$
- gen. 1: factors whose parents are of gen. 0
- gen. n: factors whose parents are of gen. $n-1$

$$\mu : \begin{cases} 0 \mapsto 010020100 \\ 1 \mapsto 010020100100 \\ 2 \mapsto 010020102010100 \end{cases}$$

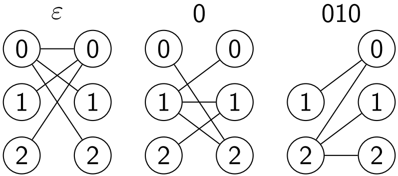


Factor generation

We can partition the factors of $\mu^\omega(0)$ as follows:

- gen. 0: factors that do not have a parent
  $=$ factors without occurrence of 0100
  $=$ internal factors of $\mu(a)0100$
- gen. 1: factors whose parents are of gen. 0
- gen. n: factors whose parents are of gen. $n-1$

$$\mu : \begin{cases} 0 \mapsto 01002\,0100 \\ 1 \mapsto 0100201\,0100 \\ 2 \mapsto 010020102010\,0100 \end{cases}$$

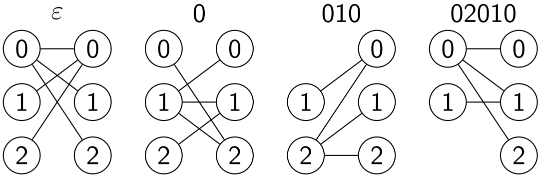


Factor generation

We can partition the factors of $\mu^\omega(0)$ as follows:

- gen. 0: factors that do not have a parent
  $=$ factors without occurrence of 0100
  $=$ internal factors of $\mu(a)0100$
- gen. 1: factors whose parents are of gen. 0
- gen. n: factors whose parents are of gen. $n-1$

$$\mu : \begin{cases} 0 \mapsto 010020100 \\ 1 \mapsto 010020100100 \\ 2 \mapsto 010020102010100 \end{cases}$$



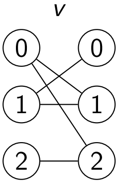
Dendric extended images

If the parent is dendric, when are the extended images dendric?

Dendric extended images

If the parent is dendric, when are the extended images dendric?

$$\mu : \begin{cases} 0 \mapsto 010020100 \\ 1 \mapsto 0100201 0100 \\ 2 \mapsto 010020102010100 \end{cases}$$

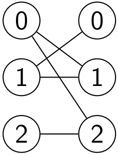


Dendric extended images

If the parent is dendric, when are the extended images dendric?

$$\mu : \begin{cases} 0 \mapsto 01002\textcolor{gray}{0100} \\ 1 \mapsto 0100201\textcolor{gray}{0100} \\ 2 \mapsto 01002010201\textcolor{gray}{0100} \end{cases}$$

v

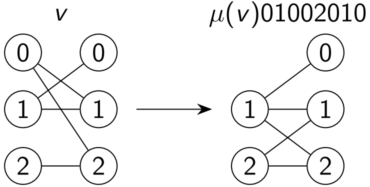


$\mu(v)01002010$

Dendric extended images

If the parent is dendric, when are the extended images dendric?

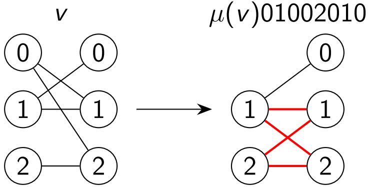
$$\mu : \begin{cases} 0 \mapsto 01002\textcolor{gray}{0100} \\ 1 \mapsto 0100201\textcolor{gray}{0100} \\ 2 \mapsto 01002010201\textcolor{gray}{0100} \end{cases}$$



Dendric extended images

If the parent is dendric, when are the extended images dendric?

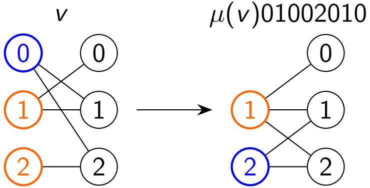
$$\mu : \begin{cases} 0 \mapsto 010020100 \\ 1 \mapsto 010020100100 \\ 2 \mapsto 0100201020100100 \end{cases}$$



Dendric extended images

If the parent is dendric, when are the extended images dendric?

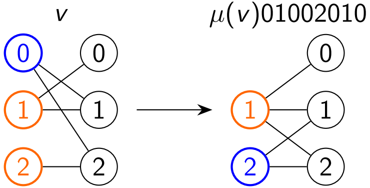
$$\mu : \begin{cases} 0 \mapsto 0100\textcolor{blue}{2}\textcolor{gray}{0100} \\ 1 \mapsto 010020\textcolor{orange}{1}\textcolor{gray}{0100} \\ 2 \mapsto 0100201020\textcolor{orange}{1}\textcolor{gray}{0100} \end{cases}$$



Dendric extended images

If the parent is dendric, when are the extended images dendric?

$$\mu : \begin{cases} 0 \mapsto 010020100 \\ 1 \mapsto 01002010100 \\ 2 \mapsto 010020102010100 \end{cases}$$



Proposition

If v is dendric, then its extended images are dendric if and only if, for any word s

- *the left letters whose images end in s form a "connected subgraph" in $\mathcal{E}(v)$;*
- *the right letters whose images (followed by 0100) start with s form a "connected subgraph" in $\mathcal{E}(v)$.*

Some graphs

Definition

For $n \in \mathbb{N}$, we define the graph G_n^L where

- the vertices are the letters;
- we have the edge (a, b) if, for all v of gen $\leq n$, if a and b are left letters in $\mathcal{E}(v)$, they are connected by a length-2 path.

Some graphs

Definition

For $n \in \mathbb{N}$, we define the graph G_n^L where

- the vertices are the letters;
- we have the edge (a, b) if, for all v of gen $\leq n$, if a and b are left letters in $\mathcal{E}(v)$, they are connected by a length-2 path.

Proposition

If the words of gen $\leq n$ are dendric, then

1. *the words of gen $n+1$ are dendric iff, for any word s,*
   - *letters whose images end in s form a connected subgraph of G_n^L*
   - *letters whose images (followed by 0100) start with s form a connected subgraph of G_n^R;*

Some graphs

Definition

For $n \in \mathbb{N}$, we define the graph G_n^L where

- the vertices are the letters;
- we have the edge (a, b) if, for all v of gen $\leq n$, if a and b are left letters in $\mathcal{E}(v)$, they are connected by a length-2 path.

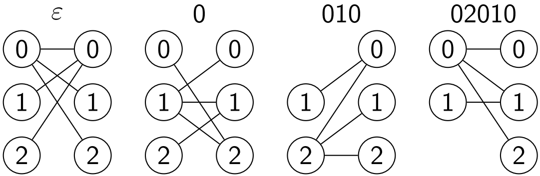
Proposition

If the words of gen $\leq n$ are dendric, then

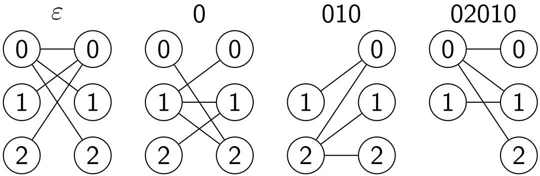
1. *the words of gen $n+1$ are dendric iff, for any word s,*
   - *letters whose images end in s form a connected subgraph of G_n^L*
   - *letters whose images (followed by 0100) start with s form a connected subgraph of G_n^R;*
2. *we can then construct G_{n+1}^L and G_{n+1}^R based on G_n^L and G_n^R.*

Back to the example

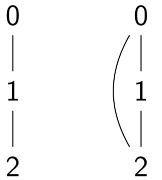
$$\mu : \begin{cases} 0 \mapsto 010020100 \\ 1 \mapsto 010020100100 \\ 2 \mapsto 010020102010100 \end{cases}$$



$$\mu : \begin{cases} 0 \mapsto 010020100 \\ 1 \mapsto 010020100100 \\ 2 \mapsto 0100201020100100 \end{cases}$$

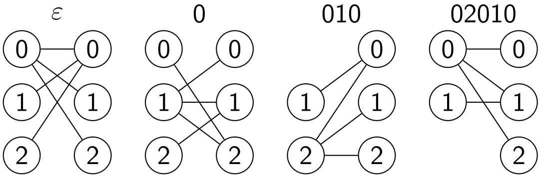


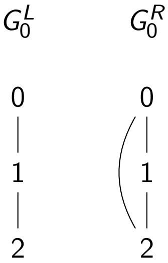
G_0^L G_0^R



Back to the example

$$\mu : \begin{cases} 0 \mapsto 01002\color{gray}{0100} \\ 1 \mapsto 0100201\color{gray}{0100} \\ 2 \mapsto 01002010201\color{gray}{0100} \end{cases}$$



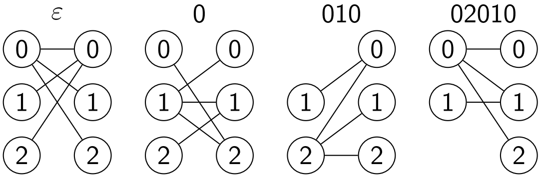


G_n^L: $\{0,1,2\}$ and $\{1,2\}$ are connected subgraphs?

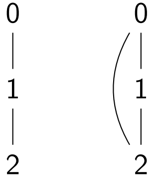
G_n^R: $\{0,1,2\}$ is a connected subgraph?

Back to the example

$$\mu : \begin{cases} 0 \mapsto 010020100 \\ 1 \mapsto 010020100100 \\ 2 \mapsto 010020102010100 \end{cases}$$



G_0^L G_0^R



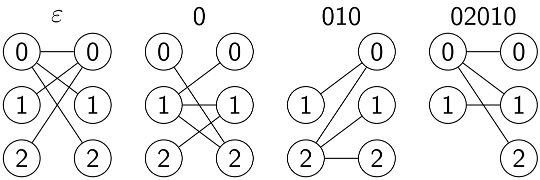
G_n^L: $\{0,1,2\}$ and $\{1,2\}$ are connected subgraphs?

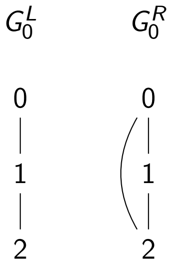
G_n^R: $\{0,1,2\}$ is a connected subgraph?

✓

Back to the example

$$\mu : \begin{cases} 0 \mapsto 010020100 \\ 1 \mapsto 010020100100 \\ 2 \mapsto 010020102010100 \end{cases}$$

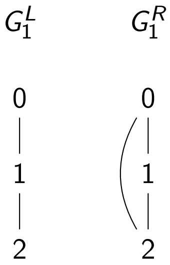




G_n^L: $\{0,1,2\}$ and $\{1,2\}$ are connected subgraphs?

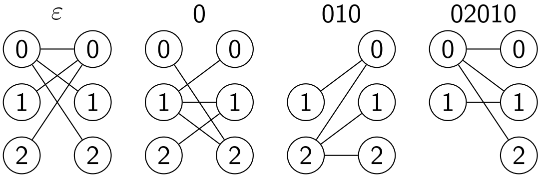
G_n^R: $\{0,1,2\}$ is a connected subgraph?

✓

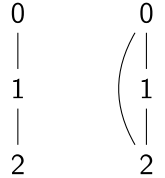


Back to the example

$$\mu : \begin{cases} 0 \mapsto 010020100 \\ 1 \mapsto 0100201 0100 \\ 2 \mapsto 0100201020 10100 \end{cases}$$



G_0^L G_0^R

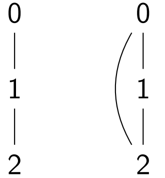


G_n^L: $\{0,1,2\}$ and $\{1,2\}$ are connected subgraphs?

G_n^R: $\{0,1,2\}$ is a connected subgraph?

✓

G_1^L G_1^R



$$\Rightarrow \mu^\omega(0) \text{ is dendric}$$

Back to the example (2)

So $\mathbf{z} = \boldsymbol{\lambda}(\boldsymbol{\mu}^{\omega}(0))$ is eventually dendric. Is it dendric?

Back to the example (2)

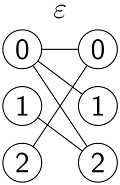
So $\mathbf{z} = \boldsymbol{\lambda}(\boldsymbol{\mu}^{\omega}(0))$ is eventually dendric. Is it dendric?

factors without parent: $\boldsymbol{\lambda}\colon \begin{cases} 0 \mapsto 020 \\ 1 \mapsto 00 \\ 2 \mapsto 0120 \end{cases}$

Back to the example (2)

So $\mathbf{z} = \boldsymbol{\lambda}(\boldsymbol{\mu}^{\omega}(0))$ is eventually dendric. Is it dendric?

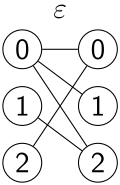
factors without parent: $\boldsymbol{\lambda}: \begin{cases} 0 \mapsto 020 \\ 1 \mapsto 00 \\ 2 \mapsto 0120 \end{cases}$



Back to the example (2)

So $\mathbf{z} = \boldsymbol{\lambda}(\boldsymbol{\mu}^{\omega}(0))$ is eventually dendric. Is it dendric?

factors without parent: $\boldsymbol{\lambda} : \begin{cases} 0 \mapsto 020 \\ 1 \mapsto 00 \\ 2 \mapsto 0120 \end{cases}$

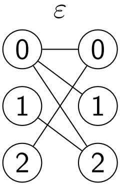


factors with parent:

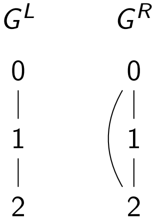
Back to the example (2)

So $\mathbf{z} = \boldsymbol{\lambda}(\mu^\omega(0))$ is eventually dendric. Is it dendric?

factors without parent: $\boldsymbol{\lambda} : \begin{cases} 0 \mapsto 02{\color{gray}0} \\ 1 \mapsto 0{\color{gray}0} \\ 2 \mapsto 012{\color{gray}0} \end{cases}$



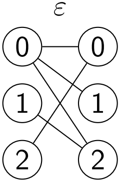
factors with parent:



Back to the example (2)

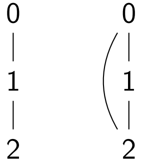
So $\mathbf{z} = \boldsymbol{\lambda}(\boldsymbol{\mu}^{\omega}(0))$ is eventually dendric. Is it dendric?

factors without parent: $\boldsymbol{\lambda}: \begin{cases} 0 \mapsto 020 \\ 1 \mapsto 00 \\ 2 \mapsto 0120 \end{cases}$



factors with parent:

G^L G^R



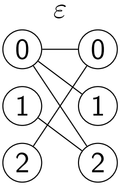
G^L: $\{0, 1, 2\}$ and $\{0, 2\}$ are connected subgraphs?

G^R: $\{0, 1, 2\}$ is a connected subgraph?

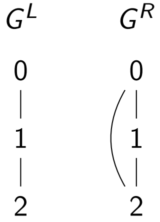
Back to the example (2)

So $\mathbf{z} = \boldsymbol{\lambda}(\mu^\omega(0))$ is eventually dendric. Is it dendric?

factors without parent: $\boldsymbol{\lambda} : \begin{cases} 0 \mapsto 020 \\ 1 \mapsto 00 \\ 2 \mapsto 0120 \end{cases}$



factors with parent:



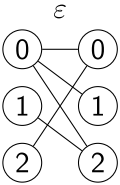
G^L: $\{0, 1, 2\}$ and $\{0, 2\}$ are connected subgraphs?

G^R: $\{0, 1, 2\}$ is a connected subgraph?

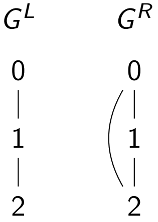
Back to the example (2)

So $\mathbf{z} = \boldsymbol{\lambda}(\mu^{\omega}(0))$ is eventually dendric. Is it dendric?

factors without parent: $\boldsymbol{\lambda}: \begin{cases} 0 \mapsto 020 \\ 1 \mapsto 00 \\ 2 \mapsto 0120 \end{cases}$



factors with parent:



G^L: $\{0, 1, 2\}$ and $\{0, 2\}$ are connected subgraphs?

G^R: $\{0, 1, 2\}$ is a connected subgraph?

$\times$

$\Rightarrow$ $\mathbf{z}$ is not dendric

Generating examples

Dendric:

- apply an appropriate return morphism on a dendric example for which you know the graphs G^L and G^R
- use the S-adic characterization (based on similar construction)

Generating examples

Dendric:
- apply an appropriate return morphism on a dendric example for which you know the graphs G^L and G^R
- use the S-adic characterization (based on similar construction)

Ev. dendric: apply a return morphism to any eventually dendric example

Generating examples

Dendric:
- apply an appropriate return morphism on a dendric example for which you know the graphs G^L and G^R
- use the S-adic characterization (based on similar construction)

Ev. dendric: apply a return morphism to any eventually dendric example

Non-ev dend: apply a return morphism to any non-eventually dendric example

Generating examples

Dendric:
- apply an appropriate return morphism on a dendric example for which you know the graphs G^L and G^R
- use the S-adic characterization (based on similar construction)

Ev. dendric: apply a return morphism to any eventually dendric example

Non-ev dend: apply a return morphism to any non-eventually dendric example

Conclusion

- Get an algorithmic procedure of the preliminary work for Durand's construction

Conclusion

- Get an algorithmic procedure of the preliminary work for Durand's construction
- Use a similar construction to decide other properties related to bispecial words

Conclusion

- Get an algorithmic procedure of the preliminary work for Durand's construction
- Use a similar construction to decide other properties related to bispecial words

Thank you for your attention!