The link between return words and extensions of factors

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LAMFA

Some notations

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- a two-sided infinite sequence x

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- a two-sided infinite sequence x
- over the alphabet $\mathcal{A}$ (minimal)
- where its language is denoted $\mathcal{L}(x)$
- and $\mathcal{L}_n(x)$ is the set of length-n words in x.

1. The protagonists

Extensions

For any $w \in \mathcal{L}(x)$,

- its *left extensions* are the letters in

$$\mathrm{E}_x^L(w) = \{a \in \mathcal{A} : aw \in \mathcal{L}(x)\},$$

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- its *bi-extensions* are the pairs of letters in

$$\mathrm{E}_x(w) = \{(a, b) \in \mathcal{A}^2 : awb \in \mathcal{L}(x)\}.$$

Extensions and factor complexity

Definition

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- $m_x(w) = \#\mathrm{E}_x(w) - \#\mathrm{E}_x^L(w) - \#\mathrm{E}_x^R(w) + 1$ is the *multiplicity* of $w \in \mathcal{L}(x)$.

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Proposition (Cassaigne)

For all n,

$$s_x(n+1) - s_x(n) = \sum_{w \in \mathcal{L}_n(x)} m_x(w).$$

Return words

$$x = \cdots 0011002001100220011000 11\cdots$$

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Definition

A *return word* for w is a word u such that

$$uw \in \mathcal{L}(x) \cap w\mathcal{A}^+ \setminus \mathcal{A}^+ w \mathcal{A}^+.$$

The *set of return words* for w is denoted $\mathcal{R}_x(w)$.

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Remark: $\mathcal{R}_x(\varepsilon) = \mathcal{A}$

Decomposition and derivation

$$x = \quad \cdots \quad 0011 \quad 002 \quad 0011 \quad 0022 \quad 0011 \quad 0 \quad 0011 \cdots$$

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$$
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 & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
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Definition

The derived sequence of x with respect to w is the sequence $D_w(x) \in \mathcal{B}^{\mathbb{Z}}$ such that $x = \theta(D_w(x))$ for a morphism θ defining a bijection between $\mathcal{B}$ and $\mathcal{R}_x(w)$.

Comparison

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- letters or pairs of letters

Return words

- (long) words

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- letters or pairs of letters
- very local
- gives the complexity of x
- used to define or characterize famous families of sequences

Return words

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- used for S-adic representations and critical exponents

First observation

Knowing the return words for w,

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$$\#\mathcal{R}_x(w) \geq \max\{\#\mathrm{E}_x^L(w), \#\mathrm{E}_x^R(w)\};$$

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Knowing the extensions of w,

- we can't know the return words for w.

2. Number of extensions and return words

A first result

Theorem (Vuillon)

x is Sturmian if and only if $\#\mathcal{R}_x(w) = 2$ for all $w \in \mathcal{L}(x)$.

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- $\#\mathcal{R}_x(w) = 2$ for all $w \in \mathcal{L}(x) \implies \#\mathcal{R}_{D_a(x)}(u) = 2$ for all $u \in \mathcal{L}(D_w(x))$

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- S-adic characterization of Sturmian sequences

A tree to count return words

For the Thue-Morse sequence,

01

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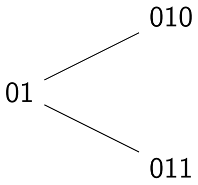

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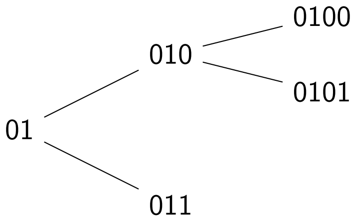

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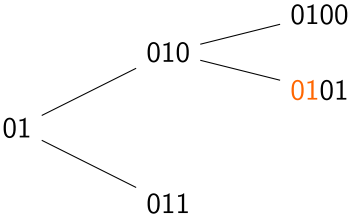

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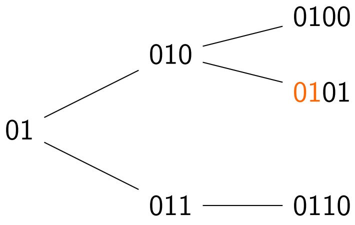

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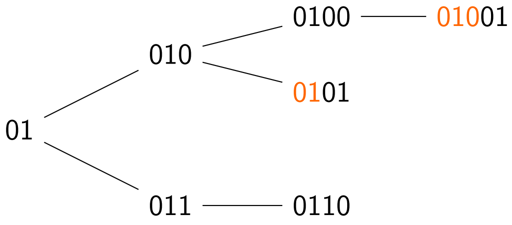

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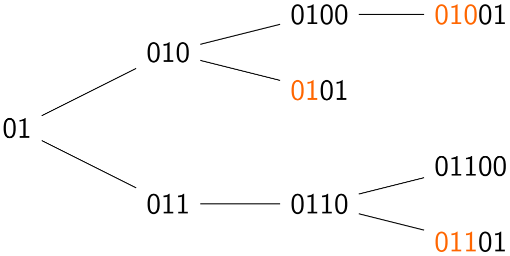

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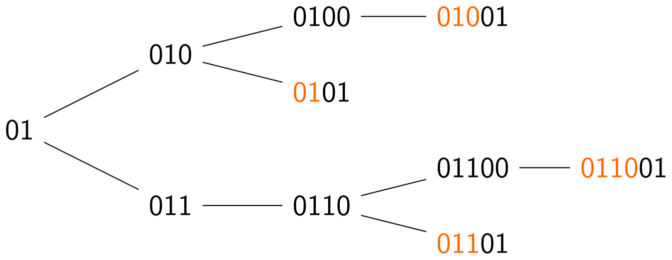

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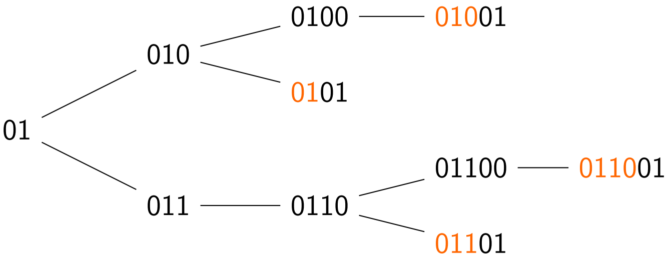

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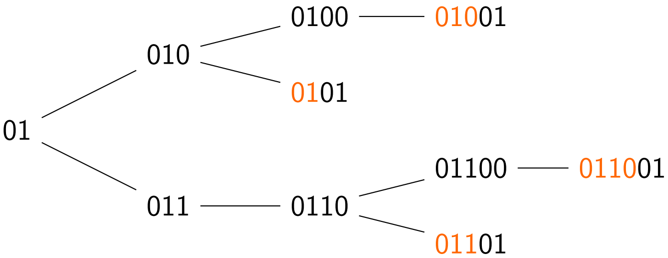

\# return words = # branches

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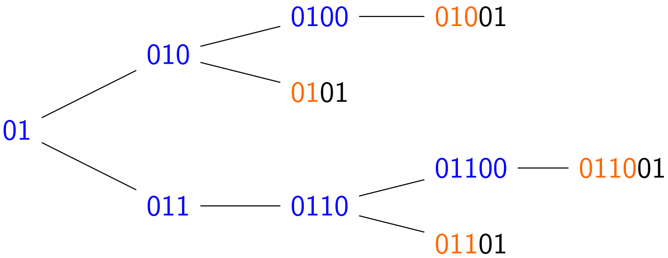

$$\begin{aligned}\#\text{ return words} &= \#\text{ branches}\\ &= 1 + \sum_{u \in S} \left(\# \mathrm{E}_x^R(u) - 1 \right)\end{aligned}$$

General case

Proposition (Balková, Pelantová, Steiner)

If x is uniformly recurrent, then for every $w \in \mathcal{L}(x)$,

$$\#\mathcal{R}_x(w) = 1 + \sum_{u \in S_w} \left(\#\mathrm{E}_x^R(u) - 1\right)$$

where $S_w = \{u \in \mathcal{L}(x) \cap w\mathcal{A}^* : |u|_w = 1\}$

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where $S_w = \{u \in \mathcal{L}(x) \cap w\mathcal{A}^* : |u|_w = 1\}$ is an x-maximal suffix code.

Definition

A set $S \subseteq \mathcal{L}(x)$ is an *x-maximal suffix code* if

- for all $u, v \in S$, if $u \in \mathrm{Suff}(v)$, then $u = v$ (suffix code)
- for each $v \in \mathcal{L}(x)$, there exists $u \in S$ such that $v \in \mathrm{Suff}(u)$ or $u \in \mathrm{Suff}(v)$ (x-maximal).

Working on sums

Lemma

If S is a finite x-maximal suffix code, then

$$\sum_{u \in S} \Big(\#\mathrm{E}_x^R(u) - 1 \Big) = \#\mathcal{A} - 1 + \sum_{\substack{u \in \mathcal{L}(x) \\ \mathrm{Suff}(u) \cap S = \emptyset}} m_x(u).$$

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Proposition (G.)

If x is uniformly recurrent, then for every $w \in \mathcal{L}(x)$,

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Similar sums to study the evolution of factor complexity when applying a morphism.

Return words for a set

Condition:

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Example:

$$x = \cdots 001100200110022001100011 0\cdots$$

and $W = \{00, 011\}$

$$\mathcal{R}_x(W) =$$
$$\mathcal{CR}_x(W) =$$

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Definition

The *complete return words* for W are the elements of

$$\mathcal{CR}_x(W) = \mathcal{L}(x) \cap W\mathcal{A}^+ \cap \mathcal{A}^+W \setminus \mathcal{A}^+W\mathcal{A}^+.$$

Number of return words for a set

Proposition (G.)

If x is uniformly recurrent, then for every factor code $W \subseteq \mathcal{L}(X)$,

$$\#\mathcal{CR}_x(W) = \#W + \sum_{u \in S_W} \left(\#\mathrm{E}_x^R(u) - 1\right)$$

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where $S_W = \{u \in \mathcal{L}(x) \cap W\mathcal{A}^* : |u|_W = 1\}$.

Neutrality

Definition

A word $w \in \mathcal{L}(x)$ is

- *neutral* if $m_x(w) = 0$,
- *weak* if $m_x(w) < 0$,
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Definition

A sequence x is *eventually neutral* if any long enough $w \in \mathcal{L}(x)$ is neutral.

Theorem (Balková, Pelantová, Steiner ; Dolce, Perrin)

Let x be uniformly recurrent with no weak $w \in \mathcal{L}_{\geq N}(x)$.
The following are equivalent:

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Let x be uniformly recurrent with no weak $w \in \mathcal{L}_{\geq N}(x)$.
The following are equivalent:

1. *every $w \in \mathcal{L}_{\geq N}(x)$ is neutral;*
2. *$\exists\, K$ st. $\#\mathcal{CR}_x(W) = \#W + K$ for every $W \subseteq \mathcal{L}_{\geq N}(x)$;*
3. *$\exists\, K$ st. $\#\mathcal{R}_x(w) = 1 + K$ for every $w \in \mathcal{L}_{\geq N}(x)$.*

Moreover, $K = s_x(N)$.

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$\Downarrow$:

$$\#\mathcal{CR}_x(W) = \#W + \#\mathcal{A} - 1 + \sum_{\substack{u \in \mathcal{L}(x) \\ |u|_W = 0}} m_x(u)$$

Theorem (Balková, Pelantová, Steiner ; Dolce, Perrin)

Let x be uniformly recurrent with no weak $w \in \mathcal{L}_{\geq N}(x)$.
The following are equivalent:

1. *every $w \in \mathcal{L}_{\geq N}(x)$ is neutral;*
2. *$\exists K$ st. $\#\mathcal{CR}_x(W) = \#W + K$ for every $W \subseteq \mathcal{L}_{\geq N}(x)$;*
3. *$\exists K$ st. $\#\mathcal{R}_x(w) = 1 + K$ for every $w \in \mathcal{L}_{\geq N}(x)$.*

Moreover, $K = s_x(N)$.

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$$\#\mathcal{CR}_x(W) = \#W + \#\mathcal{A} - 1 + \sum_{\substack{u \in \mathcal{L}(x) \\ |u|_W = 0}} m_x(u)$$

$\Uparrow$: if $m_x(v) > 0$, then $\#\mathcal{R}_x(v) < \#\mathcal{R}_x(va)$ for $a \in \mathrm{E}^R_x(v)$

Low number of return words

Proposition (adaptation of Balková, Pelantová, Steiner)

Let x be uniformly recurrent.

1. $\#\mathcal{R}_x(w) = 1$ *for every long enough* w
$\iff$ x *is ev. neutral with* $\lim_n s_x(n) = 0$ *(x is periodic).*

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Counter-example for $\#\mathcal{R}_x(w) = 4$: Thue-Morse

3. Structure of extensions and return words

Structure of return words: embedding in the free group

The free group $F_{\mathcal{A}}$ is the natural algebraic extension of $\mathcal{A}^*$ with the operations:

- inversion : $(w_1 w_2 \cdots w_n)^{-1} = w_n^{-1} \cdots w_2^{-1} w_1^{-1}$

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Free and generating sets

Definition

A set $S \subseteq F_{\mathcal{A}}$ is

- *free* if $s_1^{\eta_1} \cdots s_n^{\eta_n} \neq \varepsilon$ for any choice of $n \geq 1$, $s_1, \ldots, s_n \in S$, and of $\eta_1, \ldots, \eta_n \in \{1, -1\}$ such that $\eta_i = \eta_{i+1}$ if $s_i = s_{i+1}$;

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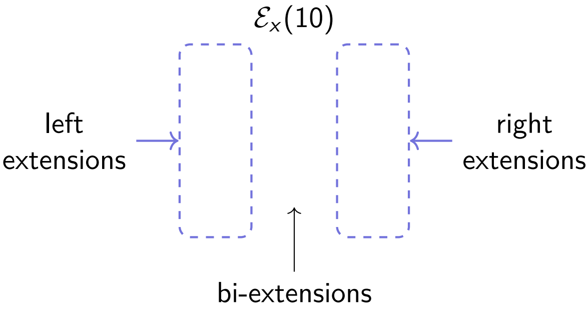
but not free:

$$(01)(0)^{-1}(01)(011)^{-1} = \varepsilon.$$

Structure of extensions: extension graph

$$x = \cdots 10010011001001001101100 \cdots$$



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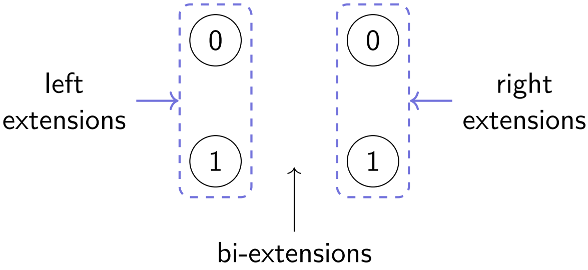
$\mathcal{E}_x(10)$



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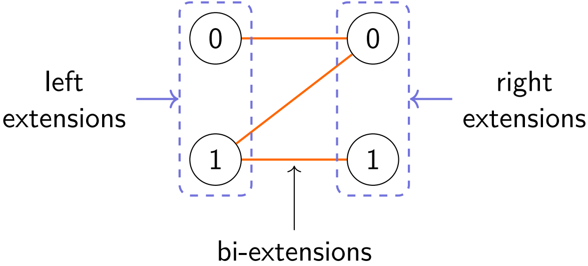

Link with Rauzy graphs

Definition

The *Rauzy graph of order n* is the graph $\Gamma_x(n)$ such that

- the vertices are the elements of $\mathcal{L}_n(x)$;
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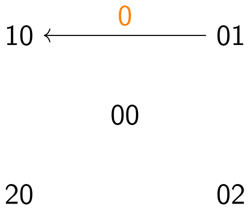

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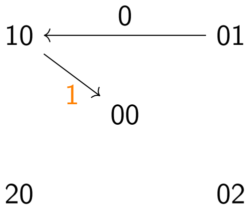

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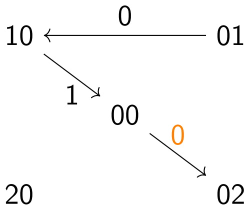

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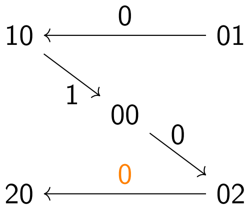

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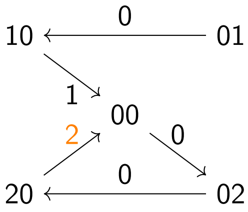

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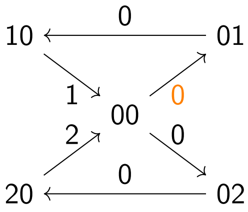

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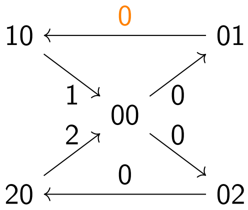

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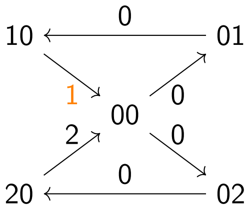

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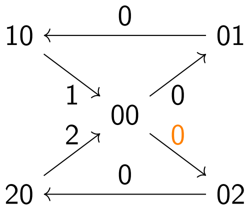

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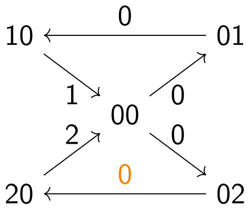

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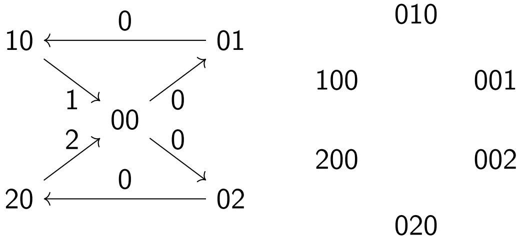

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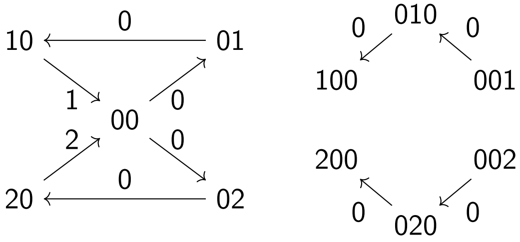

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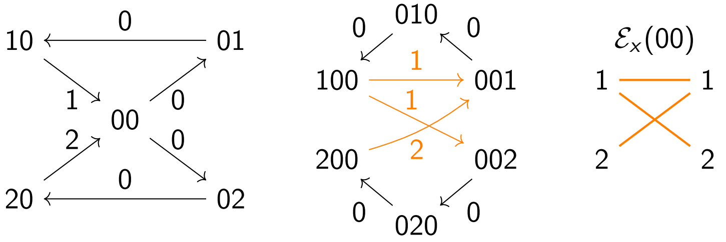

Dendricity and cie.

Definition

A word $w \in \mathcal{L}(x)$ is

- *acyclic* if $\mathcal{E}_x(w)$ is acyclic;
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Definition

A sequence x is

- *connected* if every $w \in \mathcal{L}(x)$ is connected;
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Connected extensions and generating return words

Theorem (Berthé, De Felice, Dolce, Leroy, Perrin, Reutenauer, Rindone)

If x is uniformly recurrent and connected, then $\mathcal{R}_x(w)$ generates $F_{\mathcal{A}}$ for every $w \in \mathcal{L}(x)$.

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Theorem (Goulet-Ouellet)

If x is uniformly recurrent and suffix-connected, then $\mathcal{R}_x(w)$ generates $F_{\mathcal{A}}$ for every $w \in \mathcal{L}(x)$.

Dendric extensions and basis of return words

Theorem (Berthé *et al.* & G., Goulet-Ouellet, Leroy, Stas)

Let x be uniformly recurrent. The following assertions are equivalent:

1. *x is dendric;*
2. *for every $w \in \mathcal{L}(x)$, $\mathcal{R}_x(w)$ is a basis of $F_{\mathcal{A}}$;*
3. *for every $w \in \mathcal{L}(x)$, $\mathcal{R}_x(w)$ is a tame basis of $F_{\mathcal{A}}$.*

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Proof that $1 \implies 2$:

- x connected $\implies \mathcal{R}_x(w)$ generates $F_{\mathcal{A}}$
- x dendric $\implies x$ neutral $\implies \#\mathcal{R}_x(w) = \#\mathcal{A}$

Proof that $2 \implies 1$

1. x is dendric;
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  - x connected $+$ x neutral $\implies x$ dendric

Preservation under derivation

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$\mathcal{R}_x(u)$ generates the free group over the alphabet for every u	X

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Thank you for your attention!